Math 104 Homework 1

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January 28, 2022

1 1.10

We proceed by induction. The base case is $n = 1$: we have

$$(2(1) + 1) + \cdots + (4(1) - 1) = 3 = 3(1)^2$$

Now, assume the statement is true for some $k \in \mathbb{N}$. We have

$$\begin{aligned}
&2(k+1) + 1 + 2(k+1) + 3 + \cdots + 4(k+1) - 1 \\
&= 2k + 3 + 2k + 5 + \cdots + 4k - 1 + 4k + 1 + 4k + 3 \\
&= 3k^2 + 4k + 1 + 4k + 3 - (2k + 1) \\
&= 3k^2 + 6k + 3 \\
&= 3(k^2 + 2k + 1) \\
&= 3(k+1)^2
\end{aligned}$$

This concludes the inductive step and the proof.

2 1.12

2.1 Part A

$$(a+b)^1 = a+b = \binom{1}{0}a + \binom{1}{1}b$$

$$(a+b)^2 = a^2+2ab+b^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

2.2 Part B

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \boxed{\binom{n+1}{k}}
\end{aligned}$$

2.3 Part C

We have verified the base cases $n = 1, 2, 3$ in part A. Assume that the binomial theorem holds for some $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n(a+b) \\
&= \left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n\right](a+b)
\end{aligned}$$

Now for each coefficient $a^j b^{n+1-j}$ in the resulting expression (except for $j = 0, n+1$), it can come from either $a^j b^{n-j} \cdot b$ or $a^{j-1}b^{n+1-j} \cdot a$. By the inductive hypothesis, the coefficients of the above are $\binom{n}{j}$ and $\binom{n}{j-1}$, respectively, which means their sum is $\binom{n+1}{j}$ by part B, and this is the coefficient of $a^j b^{n+1-j}$. Finally, for $j = 0$ and $n+1$, We have $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$ which is true. This completes the inductive step and the proof.

3 2.1

$\sqrt{3}$ must satisfy the equation $x^2 - 3 = 0$. By the rational root theorem, the only possible roots are [1, -1, 3, -3]. But plugging each one in, we find that

$$1^2 - 3 = -2 \neq 0$$

$$(-1)^2 - 3 = -2 \neq 0$$

$$3^2 - 3 = 6 \neq 0$$

$$(-3)^2 - 3 = 6 \neq 0$$

This means that $\sqrt{3}$ is irrational.

$\sqrt{5}$ must satisfy the equation $x^2 - 5 = 0$. By the rational root theorem, the only possible roots are [1, -1, 5, -5]. But plugging each one in, we find that

$$1^2 - 5 = -4 \neq 0$$

$$(-1)^2 - 5 = -4 \neq 0$$

$$5^2 - 5 = 20 \neq 0$$

$$(-5)^2 - 5 = 20 \neq 0$$

This means that $\sqrt{5}$ is irrational.

$\sqrt{7}$ must satisfy the equation $x^2 - 7 = 0$ By the rational root theorem, the only possible roots are [1, -1, 7, -7] But plugging each one in, we find that

$$1^2 - 7 = -6 \neq 0$$

$$(-1)^2 - 7 = -6 \neq 0$$

$$7^2 - 7 = 42 \neq 0$$

$$(-7)^2 - 7 = 42 \neq 0$$

This means that $\sqrt{7}$ is irrational.

$\sqrt{24}$ must satisfy the equation $x^2 = 24$. By the rational root theorem, the only possible roots are [1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 8, -8, 12, -12, 24, -24]. But plugging each one in, we find that

$$1^2 - 24 = -23 \neq 0$$

$$(-1)^2 - 24 = -23 \neq 0$$

$$2^2 - 24 = -20 \neq 0$$

$$(-2)^2 - 24 = -20 \neq 0$$

$$3^2 - 24 = -15 \neq 0$$

$$(-3)^2 - 24 = -15 \neq 0$$

$$4^2 - 24 = -8 \neq 0$$

$$(-4)^2 - 24 = -8 \neq 0$$
$$6^2 - 24 = 12 \neq 0$$
$$(-6)^2 - 24 = 12 \neq 0$$
$$8^2 - 24 = 40 \neq 0$$
$$(-8)^2 - 24 = 40 \neq 0$$
$$12^2 - 24 = 120 \neq 0$$
$$(-12)^2 - 24 = 120 \neq 0$$
$$24^2 - 24 = 552 \neq 0$$
$$(-24)^2 - 24 = 552 \neq 0$$

This means that $\sqrt{24}$ is irrational.

$\sqrt{31}$ must satisfy the equation $x^2 = 31$. By the rational root theorem, the only possible roots are [1, -1, 31, -31]. But plugging each one in, we find that

$$1^2 - 31 = -30 \neq 0$$
$$(-1)^2 - 31 = -30 \neq 0$$
$$31^2 - 31 = 930 \neq 0$$
$$(-31)^2 - 31 = 930 \neq 0$$

This means that $\sqrt{31}$ is irrational.

4 2.2

$\sqrt[3]{2}$ must satisfy the equation $x^3 - 2 = 0$ By the rational root theorem, the only possible roots are [1, -1, 2, -2] But plugging each one in, we find that

$$1^3 - 2 = -1 \neq 0$$

$$(-1)^3 - 2 = -3 \neq 0$$

$$2^3 - 2 = 6 \neq 0$$

$$(-2)^3 - 2 = -10 \neq 0$$

This means that $\sqrt[3]{2}$ is irrational.

$\sqrt[7]{5}$ must satisfy the equation $x^7 - 5 = 0$ By the rational root theorem, the only possible roots are [1, -1, 5, -5] But plugging each one in, we find that

$$1^7 - 5 = -4 \neq 0$$

$$(-1)^7 - 5 = -6 \neq 0$$

$$5^7 - 5 = 78120 \neq 0$$

$$(-5)^7 - 5 = -78130 \neq 0$$

This means that $\sqrt[7]{5}$ is irrational.

$\sqrt[4]{13}$ must satisfy the equation $x^4 - 13 = 0$ By the rational root theorem, the only possible roots are [1, -1, 13, -13] But plugging each one in, we find that

$$1^4 - 13 = -12 \neq 0$$

$$(-1)^4 - 13 = -12 \neq 0$$

$$13^4 - 13 = 28548 \neq 0$$

$$(-13)^4 - 13 = 28548 \neq 0$$

This means that $\sqrt[4]{13}$ is irrational.

5 2.7

5.1 Part A

Notice that $4+2\sqrt{3}=1+2\sqrt{3}+(\sqrt{3})^2=(1+\sqrt{3})^2$. Therefore,

$$\sqrt{4+2\sqrt{3}}-\sqrt{3}=1+\sqrt{3}-\sqrt{3}=\boxed{1}$$

5.2 Part B

Notice that $6+4\sqrt{2}=4+4\sqrt{2}+(\sqrt{2})^2=(2+\sqrt{2})^2$. Therefore,

$$\sqrt{6+4\sqrt{2}}-\sqrt{2}=2+\sqrt{2}-\sqrt{2}=\boxed{2}$$

6 3.6

6.1 Part A

We first apply the triangle inequality to $a + b$ and c to get

$$|a + b + c| \leq |a + b| + |c|$$

Next, we apply the triangle inequality to a and b on the right hand side of the above equation to get

$$|a + b| + |c| \leq |a| + |b| + |c|$$

Putting these two inequalities together, we find that

$$|a + b + c| \leq |a| + |b| + |c|$$

as desired.

6.2 Part B

We have proved the base case in part A. Assume the statement holds for an integer k. Now, we have

$$\begin{aligned} |a_1 + \cdots + a_k + a_{k+1}| &\leq |a_1 + \cdots + a_k| + |a_{k+1}| \\ &\leq |a_1| + |a_2| + \cdots + |a_k| + |a_{k+1}| \end{aligned}$$

by the inductive hypothesis. This completes the inductive step and the proof.

7 4.11

We know that $a < a + \frac{b-a}{k} < b$ and $a + \frac{b-a}{k}$ is a rational number for any integer $k \geq 2$. Since there are infinitely many integers, there must be an infinite amount of rational numbers between a and b.

8 4.14

8.1 Part A

First we show $\sup(A) + \sup(B)$ is a valid upper bound. For any $a \in A$ and $b \in B$, we have

$$a + b \leq \sup(A) + b \leq \sup(A) + \sup(B)$$

Next, we show that it is the least upper bound. Assume there is a lower upper bound, so that we can write it as $\sup(A) + \sup(B) - 2\epsilon$, where $\epsilon > 0$. But we know that there exists a $a \in A$ such that $a > \sup(A) - \epsilon$. Likewise, we know that there exists a $b \in B$ such that $b > \sup(B) - \epsilon$. Adding these two together yields

$$a + b > \sup(A) + \sup(B) - 2\epsilon$$

showing that the lower upper bound is not valid. Therefore, $\sup(A + B) = \sup(A) + \sup(B)$, as desired.

8.2 Part B

First we show $\inf(A) + \inf(B)$ is a valid lower bound. For any $a \in A$ and $b \in B$, we have

$$a + b \geq \inf(A) + b \geq \inf(A) + \inf(B)$$

Next, we show that it is the greatest lower bound. Assume there is a greater lower bound, so that we can write it as $\inf(A) + \inf(B) + 2\epsilon$, where $\epsilon > 0$. But we know that there exists a $a \in A$ such that $a < \inf(A) + \epsilon$. Likewise, we know that there exists a $b \in B$ such that $b < \inf(B) + \epsilon$. Adding these two together yields

$$a + b < \inf(A) + \inf(B) + 2\epsilon$$

showing that the greater lower bound is not valid. Therefore, $\inf(A + B) = \inf(A) + \inf(B)$, as desired.

9 7.5

9.1 Part A

We know that

$$(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n) = n^2+1-n^2 = 1$$

which means we can rewrite

$$s_n = \frac{1}{\sqrt{n^2+1}+n}$$

But we know that

$$\lim \sqrt{n^2+1}+n = \infty$$

so

$$\lim s_n = \boxed{0}$$

9.2 Part B

We know that

$$(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n) = n^2+n-n^2 = n$$

which means we can rewrite

$$s_n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

Since $\lim \frac{1}{n} = 0$, we have

$$\lim s_n = \frac{1}{\sqrt{1}+1} = \boxed{\frac{1}{2}}$$

9.3 Part C

We know that

$$(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n) = 4n^2+n-4n^2 = n$$

which means we can rewrite

$$s_n = \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+\frac{1}{n}}+2}$$

Since $\lim \frac{1}{n} = 0$, we have

$$\lim s_n = \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$