Math 104 Homework 10

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April 22, 2022

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1 Q 1: Ross 33.4

Let

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

Here $f(x)$ is not integral, but $|f(x)| = 1$ which is integrable.

2 Q 2: Ross 33.7

2.1 Part A

$$\begin{aligned} U(f^2,P) - L(f^2,P) &= \sum \sup x_i^2 - \inf x_i^2 \\ &\leq \sum M_i^2 - m_i^2 \\ &= \sum (M_i - m_i)(M_i + m_i) \\ &\leq \sum 2B(M_i - m_i) \\ &= 2B(U(f,P) - L(f,P)) \end{aligned}$$

2.2 Part B

If f is integrable this means that $U(f,P) = L(f,P)$. But this implies that $U(f^2,P) - L(f^2,P) \leq 0$. Since $U(f^2,P) \geq L(f^2,P)$, this implies that they are equal, which shows that f^2 is integrable as well.

3 Q 3: Ross 33.13

Suppose not, for the sake of contradiction. Then, $f(x) > g(x) \forall x \in [a, b]$ (without loss of generality; otherwise just switch f and g). However, then we have

$$U(f, P) = \sum_P \sup f > \sum_P \sup g = U(g, P)$$

This shows that the two integrals are actually not equal, a contradiction.

4 Q 4: Ross 35.4

4.1 Part A

We know

$$\frac{dF(x)}{dx} = \cos x$$

so

$$dF(x) = \cos x \, dx$$

Then,

$$\int_0^{\pi/2} x \, dF(x) = \int_0^{\pi/2} x \cos x \, dx = g(\pi/2) - g(0)$$

where

$$g(x) = x \sin x - \cos x$$

This evaluates to

$$g(\pi/2) - g(0) = \boxed{\frac{\pi}{2} - 1}$$

4.2 Part B

We know

$$\frac{dF(x)}{dx} = \cos x$$

so

$$dF(x) = \cos x \, dx$$

Then,

$$\int_{-\pi/2}^{\pi/2} x \, dF(x) = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = g(\pi/2) - g(-\pi/2)$$

where

$$g(x) = x \sin x - \cos x$$

This evaluates to

$$g(\pi/2) - g(-\pi/2) = \boxed{0}$$

5 Q 5: Ross 35.9a

Let $g(x) = \inf_{[a,b]} f$, so g is a constant function. Then, $\int_a^b g\,dF$ lower bounds $\int_a^b f\,dF$. Similarly, if $h(x) = \sup_{[a,b]} f$ then $\int_a^b h\,dF$ upper bounds $\int_a^b f\,dF$. But since g and h are constants, the bounding intervals evaluate to $g(F(b)-F(a))$ and $h(F(b)-F(a))$ respectively. By intermediate value theorem, there exists a $x \in [a, b]$ such that $f(x)(F(b) - F(a))$ is the actual value of the integral.