Math 104 Homework 2

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1 Ross 9.9

1.1 Part A

We proceed by contradiction. Assume $\lim t_n$ is finite. Then, consider $\epsilon = 1$. We know that there exists an integer N such that $t_n < \lim t_n + \epsilon = \lim t_n + 1$ for all $n > N$. However, we also know that there exists an integer M such that $s_m > \lim t_n + 1$ for all $m > M$. Let $z = \max(M, N)$. But this means that $s_z > t_z$! This violates the assumption that $s_n \leq t_n$. This means that we must have $\lim t_n = +\infty$.

1.2 Part B

We proceed by contradiction. Assume $\lim s_n$ is finite. Then, consider $\epsilon = 1$. We know that there exists an integer M such that $s_n > \lim t_n - \epsilon = \lim t_n - 1$ for all $n > N$. However, we also know that there exists an integer N such that $t_n < \lim s_n - 1$ for all $n > N$. Let $z = \max(M, N)$. But this means that $s_z > t_z$! This violates the assumption that $s_n \leq t_n$. This means that we must have $\lim s_n = -\infty$.

1.3 Part C

Let $S = \lim s_n$ and $T = \lim t_n$. We proceed by contradiction. Assume $\lim s_n > \lim t_n$. Then, consider any $\epsilon > 0$ such that $\epsilon < 0.5(S - T)$. This means that there exists an integer M such that $s_m > s - \epsilon$ for all $m \geq M$. Likewise, there exists an integer N such that $t_n < t + \epsilon$ for all $n \geq N$. However, let $z = \max(M, N)$. We have

$$s_z > s - \epsilon > s - 0.5(S - T) = 0.5T - 0.5S = T + \epsilon > t_z$$

which violates the assumption that $s_n \leq t_n$. Thus, we must have $\lim s_n \leq \lim t_n$, as desired.

2 Ross 9.15

Let b be any integer greater than a. Notice that $\frac{a^n}{n!} > 0$ for all positive integer n, so the limit is non-negative. Then, for $n > b$, we have

$$\lim \frac{a^n}{n!} = \lim a^b \frac{a^{n-b}}{n!} = a^b \lim \frac{a^{n-b}}{n!} \leq a^b \lim \frac{a^{n-b}}{b^{n-b}} = a^b \lim \left(\frac{a}{b}\right)^{n-b} = 0$$

since $a < b$. Therefore, we have shown that $\lim \frac{a^n}{n!} = 0$ as desired.

3 Ross 10.7

Consider the sequence $a_n = (1, 0.104, 0.010816, 0.001124864, \dots)$ so $a_n = 0.104^n$ (let this be 0-indexed). We know that $\lim a_n = 0$. For every a_n, we also know that there exists a $s \in S$ such that $s > \sup S - a_n$. Let $s_n =$ that s for all values of n.

Now, I show that $\lim s_n = \sup S$. Consider any $\epsilon > 0$. We know that there exists a positive integer N such that $a_n < \epsilon$ for all $n \geq N$. However, this means that $s_n > \sup S - a_n > \sup S - \epsilon$ for all $n \geq N$. Since $s_n < \sup S$, this proves that $\lim s_n = \sup S$ as desired.

4 Ross 10.8

We want to show that $\sigma_n > \sigma_{n-1}$ for all integer $n > 1$. Instead, we show that $n\sigma_n > n\sigma_{n-1}$. We have

$$\begin{aligned}
n\sigma_n &= s_1 + s_2 + \cdots + s_n \\
&= \frac{n}{n-1}s_1 - \frac{1}{n-1}s_1 + \frac{n}{n-1}s_2 - \frac{1}{s-1}s_2 + \cdots + \frac{n}{n-1}s_{n-1} - \frac{1}{s-1}s_{n-1} + s_n \\
&= \frac{n}{n-1}(s_1 + s_2 + \cdots + s_{n-1}) + \frac{1}{n-1}(s_n - s_1) + \frac{1}{n-1}(s_n - s_2) + \cdots + \frac{1}{n-1}(s_n - s_{n-1}) \\
&> n\sigma_{n-1}
\end{aligned}$$

5 Ross 10.9

5.1 Part A

$$s_2 = \frac{1}{2}(1)^2 = \frac{1}{2}$$
$$s_3 = \frac{2}{3}\left(\frac{1}{2}\right)^2 = \frac{1}{6}$$
$$s_4 = \frac{3}{4}\left(\frac{1}{6}\right)^2 = \frac{1}{48}$$

5.2 Part B

Since $s_2, s_3, s_4 < 1$, we have for $n > 4$:

$$s_{n+1} = \frac{n}{n+1}(s_n)^2 > (s_n)^2 > s_n$$

since $s_n < 1$. This shows inductively that (a) the sequence (s_n) is decreasing and (b) $s_n < 1$. Notice that $s_2, s_3, s_4 > 0$, and both $\frac{n}{n+1}$ and s_n^2 are positive. So (s_n) is bounded below by 0. Since it is decreasing and bounded below by 0, the limit exists.

5.3 Part C

We show that $s_n \leq \frac{1}{n}$. We proceed by induction. We have

$$s_{n+1} = \frac{n}{n+1}(s_n)^2 \leq \frac{n}{n+1}s_n \leq \frac{n}{n+1}\frac{1}{n} = \frac{1}{n+1}$$

Since $\lim \frac{1}{n} = 0$, we have $\lim s_n \leq 0$. But we also have $s_n \geq 0$ for all n, so $\lim s_n \geq 0$. This means that $\lim s_n = 0$ as desired.

6 Ross 10.10

6.1 Part A

$$s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$$
$$s_3 = \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{9}$$
$$s_4 = \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{14}{27}$$

6.2 Part B

The base case is clear. For the inductive step, we have

$$s_{n+1} = \frac{1}{3}(s_n+1) > \frac{1}{3}\left(\frac{1}{2}+1\right) = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$$

which completes the induction.

6.3 Part C

By part B, we know that $1 < 2s_n$, so

$$s_{n+1} = \frac{1}{3}(s_n+1) < \frac{1}{3}(s_n+2s_n) = s_n$$

6.4 Part D

$\lim s_n$ exists because it is decreasing and bounded below by 0.5. So, let $S = \lim s_n$. We know that S must satisfy the equation

$$S = \frac{1}{3}(S+1)$$

Solving, we find that

$$2S+1$$

or

$$S = \lim s_n = \boxed{\frac{1}{2}}$$

7 Ross 10.11

7.1 Part A

First, t_n is bounded below by 0 because $1 - \frac{4}{n^2} > 0$ and $t_1 > 0$. Also, t_n is decreasing because $1 - \frac{4}{n^2} < 1$. Thus the limit exists.

7.2 Part B

Using a program we find that $t_{1000000} \approx 0.63661977340318$. We also have $\frac{2}{\pi} \approx 0.63661977238578$, so it seems to be around $\frac{2}{\pi}$.

8 Squeeze Test

By 9.9c on (b_n) and (c_n), we have $\lim b_n \leq \lim c_n = L$. By 9.9c on (a_n) and (b_n), we have $\lim b_n \geq \lim a_n = L$. These two show that $\lim b_n = L$, as desired.