Math 104 Homework 3

Jonathan Guo

February 11, 2022

Contents

1 Ross 10.6

1.1 Part A

Consider any $\epsilon > 0$. Let N be the least integer such that $\frac{1}{2^N} < \epsilon$. We will show that for all $n, m > N$ that $|s_m - s_n| < \epsilon$. Assume without loss of generality that $m > n$. Then, we have

$$\begin{aligned} |s_m - s_n| &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\ &< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots \\ &= \frac{1}{2^{n-1}} \\ &\leq \frac{1}{2^N} \\ &< \epsilon \end{aligned}$$

1.2 Part B

The result is not true, in particular because $\sum \frac{1}{n}$ diverges. For example, consider the sequence $s_n = \sum_{j=1}^{n} \frac{1}{j}$. Then, we have $s_{n+1} - s_n = \frac{1}{n+1} < \frac{1}{n}$, but $\lim s_n \to +\infty$ which is not convergent and therefore not cauchy.

2 Ross 11.2

2.1 Part A

a_n: Consider all even terms $a_2, a_4, a_6, \ldots$. This subsequence is $1, 1, 1, \ldots$ which is monotone.

b_n: b_n itself is already a monotone decreasing sequence.

c_n: c_n itself is already a monotone increasing sequence.

d_n: d_n itself is already a monotone decreasing sequence because $d_1 > \frac{6}{7}$ and each subsequent term gets closer and closer to $\frac{6}{7}$.

2.2 Part B

a_n: The set of subsequential limits is $\{-1, 1\}$.

b_n: The set of subsequential limits is $\{0\}$.

c_n: The set of subsequential limits is $\{+\infty\}$.

d_n: The set of subsequential limits is $\{\frac{6}{7}\}$.

2.3 Part C

a_n: lim inf is -1 and lim sup is 1.

b_n: lim inf is 0 and lim sup is 0.

c_n: lim inf is $+\infty$ and lim sup is $+\infty$.

d_n: lim inf is $\frac{6}{7}$ and lim sup is $\frac{6}{7}$.

2.4 Part D

a_n: This sequence does not converge but also does not diverge to $\pm\infty$.

b_n: This sequence converges to 0.

c_n: This sequence diverges to $+\infty$.

d_n: This sequence converges to $\frac{6}{7}$.

2.5 Part E

a_n: This sequence is bounded below by -1 and above by 1.

b_n: This sequence is bounded below by 0 and above by 1.

c_n: This sequence is bounded below by 1 unbounded above.

d_n: This sequence is bounded below by $\frac{6}{7}$ and above by 2.5.

3 Ross 11.3

3.1 Part A

s_n: Consider all multiple of 6 terms $a_6, a_{12}, a_{18}, \ldots$. This subsequence is $1, 1, 1, \ldots$ which is monotone.

t_n: b_n itself is already a monotone decreasing sequence.

u_n: Consider all even terms $a_2, a_4, a_6, \ldots$. This subsequence is $|\frac{1}{2^n}|$ which is monotone decreasing.

v_n: Consider all even terms $a_2, a_4, a_6, \ldots$. This subsequence is $1 + \frac{1}{n}$ which is monotone decreasing.

3.2 Part B

s_n: The set of subsequential limits is $\{-1, -0.5, 0.5, 1\}$.

t_n: The set of subsequential limits is $\{0\}$.

u_n: The set of subsequential limits is $\{0\}$.

v_n: The set of subsequential limits is $\{-1, 1\}$.

3.3 Part C

s_n: lim inf is -1 and lim sup is 1.

t_n: lim inf is 0 and lim sup is 0.

u_n: lim inf is 0 and lim sup is 0.

v_n: lim inf is -1 and lim sup is 1.

3.4 Part D

s_n: This sequence does not converge but also does not diverge to $\pm\infty$.

t_n: This sequence converges to 0.

u_n: This sequence converges to 0.

v_n: This sequence does not converge but also does not diverge to $\pm\infty$.

3.5 Part E

s_n: This sequence is bounded below by -1 and above by 1.

t_n: This sequence is bounded below by 0 and above by 0.6.

u_n: This sequence is bounded below by -0.5 and above by 0.25.

v_n: This sequence is bounded below by -1 and above by 1.5.

4 Ross 11.5

4.1 Part A

The set of subsequential limits of (q_n) is the real interval $[0, 1]$. This is because for any real number r and for any $\epsilon >)$, there are infinitely many rational numbers in the range $(r - \epsilon, r + \epsilon)$. Thus, for any $\epsilon > 0$ and $N > 0$, there exists a $n > N$ with $|q_n - r| < \epsilon$, and this shows that r is a subsequential limit.

4.2 Part B

lim sup $q_n = 1$ and lim inf $q_n = 0$

5 lim sup

lim sup is the limit of the sup of the tail of a sequence. The difference between sup is that sup takes into account the entire sequence but lim sup only cares about the start of the sequence.

For example, one misconception about lim sup could be that you can swap the places of lim and sup, yielding sup lim.