Math 104 Homework 4

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1 Ross 12.10

If $\limsup |s_n| < +\infty$, then that means $s_n \leq \limsup |s_n|$ and $s_n \geq -\limsup |s_n|$ for all n. But this means that we have found a finite upper bound and a finite lower bound for (s_n), which shows that it is bounded.

If s_n is bounded, then there exists a finite upper bound U and a finite lower bound L. But this means $\limsup |s_n| \leq \max(|U|, |L|)$, which shows that $\limsup |s_n| < +\infty$, as desired.

2 Ross 12.12

2.1 Part A

2.2 Part B

If $\lim s_n$ exists, this means that $\liminf s_n = \limsup s_n = \lim s_n$. This means that $\liminf \sigma_n = \limsup \sigma_n = \lim s_n$, so $\lim \sigma_n = \lim s_n$ as well.

2.3 Part C

Let $(s_n) = +1, -1, +1, -1, \ldots$. Then, $\lim s_n$ does not exist, but we have $(\sigma_n) = 1, 0, 1/3, 0, 1/5, \ldots$ which has $\lim \sigma_n = 0$.

3 Ross 14.2

3.1 Part A

$$\sum_n \frac{n-1}{n^2} = \sum_n \frac{1}{n} - \sum_n \frac{1}{n^2}$$

A divergent sum minus a convergent sum still diverges, so this series is divergent.

3.2 Part B

This series neither converges nor diverges because it just oscillates between -1 and 0.

3.3 Part C

This series converges because we have $\sum \frac{3n}{n^3} = 3 \sum \frac{1}{n^2}$ which converges.

3.4 Part D

This series converges by ratio test, the ratio approaches $\frac{1}{3}$.

3.5 Part E

This series converges by ratio test, the ratio approaches $\frac{1}{n}$ which approaches 0.

3.6 Part F

This series converges by comparison test to something like $\frac{1}{n^{1.1}}$.

3.7 Part G

This series converges. In fact this series sums to $\frac{0.5}{0.5^2} = 2$.

4 Ross 14.10

Consider

$$(a_n) = 2^{(-1)^n + n}$$

so $(a_n) = 1, 8, 4, 32, 16, \ldots$. By ratio test, we have

$$\liminf |a_{n+1}/a_n| = \frac{1}{2} < 1 < 8 = \limsup |a_{n+1}/a_n|$$

which gives us no information. But by root test, we have $\lim a_n^{1/n} = 2 > 1$ so this diverges.

5 Rudin 3.6

5.1 Part A

This series diverges because

$$\sum a_n = \sqrt{n+1} - \sqrt{n} + \sqrt{n} - \sqrt{n-1} + \cdots - \sqrt{2} + \sqrt{1} = \sqrt{n+1} - 1$$

which diverges as $n \to +\infty$.

5.2 Part B

5.3 Part C

5.4 Part D

6 Rudin 3.7

7 Rudin 3.9

8 Rudin 3.11

8.1 Part A

8.2 Part B

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+2}}{s_{N+2}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \frac{a_{N+2}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1} + a_{N+2} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}}\end{aligned}$$

8.3 Part C

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_n s_{n-1}} \geq \frac{a_n}{s_n^2}$$

8.4 Part D