Math 104 Homework 5

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1 Ross 13.3

1.1 Part A

- Reflexive: $d(x, x) = \sup \{|x_i - x_i|\} = \sup \{0, 0, \dots\} = 0$.
- Symmetric: $d(x, y) = \sup \{|x_j - y_j|\} = \sup \{|y_j - x_j|\} = d(y, x)$
- Triangle: $d(x, y) = \sup \{|x_j - y_j|\} \geq \sup \{|x_j - z_j| + |z_j - y_j|\} \geq \sup \{|x_j - z_j|\} + \sup \{|z_j - y_j|\} = d(x, z) + d(z, y)$

1.2 Part B

d^* satisfies $d^*(x, x) = 0$ and $d^*(x, y) = d^*(y, x)$. But some choices of x and y yield $d^* = +\infty$, like $x = 0, 0, \dots$ and $y = 1, 1, \dots$.

2 Ross 13.5

2.1 Part A

$x \in \bigcap S \setminus U : U \in \mathcal{U} \iff x \in S \wedge \forall U \in \mathcal{U} x \notin U \iff x \in S \wedge x \notin \bigcup U \iff x \in S \setminus \bigcup U$

2.2 Part B

Let $\mathcal{V}$ be the collection of closed sets, and let $\mathcal{U}$ be the collection of complements of $\mathcal{V}$, so

$$\mathcal{V} = \{S \setminus U : U \in \mathcal{U}\}$$

Note that $\mathcal{U}$ is now a collection of open sets. By part A, we have

$$\bigcap V \in \mathcal{V} = \bigcap \{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup U \in \mathcal{U}$$

Since the union of a collection of open sets is open, we have $\bigcup U \in \mathcal{U}$ is open, so $S \setminus \bigcup U \in \mathcal{U}$ is closed. But this shows that $\bigcap V \in \mathcal{V}$ is closed, as desired.

3 Ross 13.7

Let S be the open set in $\mathbb{R}$. For every point $x \in S$, we know there exists an open ball (open interval) surrounding x that is entirely in S. Now, consider the union of all of these open intervals. This is the entire S, since it contains all of the points in S. However, these intervals are not disjoint. We can make them disjoint by just unioning two intervals if they are not disjoint. So we can replace two overlapping open intervals with one open interval. Once we are done with this process, the remaining open intervals will all be disjoint, so we have proven that S is the disjoint union of a finite or infinite sequence of open intervals.

4 Q 4

Let p be something in $\bar{S_1} = S_2$. we will show it is also in S_1. So, we know there is a sequence (p_n) in S_1 that converge to p. We also know that for each p_i, there exists a sequence $(p_i)_n$ that converges to p_i. So we can pick one element from each sequence using a diagonalization argument to show that there also exists a sequence that converge to p, which shows that $p \in S_1$ as well.

5 Q 5

We know $\bar{S}$ is closed. Consider any any other closed subset in X that contains S. We will show that this is a superset of $\bar{S}$. This is because it must contains S, and all of the subsequential points as well. This means that every other closed subset in X that contains S also contains $\bar{S}$. So, the intersection of all of the closed subsets in X that contains S is the intersection of $\bar{S}$ with a bunch of supersets of $\bar{S}$ which means it is just $\bar{S}$.