Math 104 Homework 6

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1 Q 1

True. Consider any sequence of points $(a_n) = (b_n, c_n)$ in $[0,1]^2$. We know that there exists a subsequence of the (b_n) that converges to some point $x \in [0,1]$. Consider the subsequence of (a_n) that corresponds to this subsequence, so $(a_n)_k$. We know that there exists a subsequence of the (c_n) in this one that converges to some point $y \in [0,1]$. Now, since the (b_n) of this already converges, we know that any subsequence of it also converges to $x \in [0,1]$. Thus, if we take this subsequence, it will converge to $(x,y) \in [0,1]^2$, which shows that $[0,1]^2$ is sequentially compact.

2 Q 2

E is uncountable. There is a bijection between E and the set of real numbers whose decimal expansion only consists of the digits 0 and 1 by simply replacing 4 with 0 and 7 with 1. Then, there is a bijection between this set and the set of real numbers whose *binary* representation only consists of the digits 0 and 1. This set is just $[0, 1]$ which is uncountable. Therefore, E is uncountable.

However, E is sequentially compact because any sequence in E is bounded and therefore has a convergent subsequence. Also, since E is closed, the convergent point is in E, which shows that it is sequentially compact and therefore compact.

3 Q 3

If the set of A_i is finite then no. But if it is countably infinite then yes. For example, take $A_i = (\frac{1}{i}, (0,1))$ which is a subset of $\mathbb{R}^2$. Each of these is a set. We will show there is a limit point in the closure of the union of these A_is that is not in the closure of any one of the A_is. Consider the sequence $a_n = \frac{1}{n}, \frac{1}{n}$. Each a_i is in A_i. The sequence approaches $(0,0)$ so that is in $\bar{B}$. However, $(0,0)$ is in none of the A_is because the x-coordinate of any limit point of A_i must be $\frac{1}{i}$. Therefore it is possible that the inclusion is a strict inclusion.

4 Q 4

E from question 2 is a closed set, but it is not a countable union of closed sets because it is an uncountable union of closed sets.