Math 104 Homework 7

Jonathan Guo

March 18, 2022

Contents

1 Q 1 **2**

2 Q 2 **3**
- 2.1 Open 3
- 2.2 Closed 3
- 2.3 Bounded 3
- 2.4 Compact 3
- 2.5 Connected 3

3 Q 3 **4**

1 Q 1

Consider an open set cover of $X \times Y$. Now project this onto X, where the projection just means we take the x component of each element of each set. We know that there is a finite open set cover of X, so take this. Now, for each set S in this finite open set cover, consider the set of all open sets of the original open set sets that intersect S. We know that there is a finite set covering for the projection of these onto Y. If we take the set of all of these sets, it is finite (since it is finite times finite) and it covers everything in $X \times Y$. Therefore we have found a finite open set cover for $X \times Y$.

2 Q 2

2.1 Open

False. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(x)$. Consider $A = (-104, 104) \subset \mathbb{R}$ be an open subset. However, the map $f(A)$ is the not open subset $[-1, 1]$. Thus, f does not have to preserve openness.

2.2 Closed

False. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{1}{x^2+1}$. Consider $A = [0, +\infty) \subset \mathbb{R}$ be a closed subset. However, the map $f(A)$ is the not closed subset $(0, 0.5]$. Thus, f does not have to preserve closeness.

2.3 Bounded

False. From problem 3 we know there exists a surjective continuous map $f : (0, 1) \to \mathbb{R}$. We know at least one of $(0, 0.5)$ and $[0.5, 1)$ must be unbounded as one of them must contain $+\infty$. Thus, f does not have to preserve boundedness.

2.4 Compact

True. Consider an open set covering of $f(A)$. By definition, the preimage of each of these sets is an open set of A, so the preimage is a set covering of A. But we know there is a finite open set covering of A. Taking the corresponding images of those sets gives us a finite open set covering of $f(A)$, showing that $f(A)$ must also be compact.

2.5 Connected

True. We can try the contrapositive. Assume $f(A)$ not connected, so we can write it as a disjoint union of two open subsets. Look at the preimages of those two open subsets. We know they must be disjoint because a function maps one value to only one value. We also know that both of them are open as f is continuous. Finally, we know that they union to A because their image unions to $f(A)$. Thus, we have written A as a disjoint union of two open subsets, so A is not connected as well. This proves that f must preserve connectedness.

3 Q 3

Assume that such an f exists. We know that $[0, 1]$ is compact, which means that from question 2 we know that $\mathbb{R}$ must also be compact. However, this is not the case; therefore, such an f cannot exist.