Math 104 Homework 8

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April 1, 2022

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1 Q 1

We show that (f_n) converges uniformly to the function $f(x) = \frac{1}{2}$. Let $\epsilon > 0$. Then, choose N such that $\frac{N+1}{2N-1} - 0.5 < \epsilon$ and $\frac{N-1}{2N+1} - 0.5 > \epsilon$. Then, because $f_N(x)$ and $f_n(x)$ for $n > N$ will fall into those ranges, the distance from $f_n(x)$ to $f(x)$ will be less than ϵ for all x. This shows the uniform convergence.

2 Q 2

Let $f_n(x) = \sum_{k=1}^{n} a_k x^k$. Each $f_n(x)$ is continuous because it is the sum of finitely many polynomials. Each $f_n(x)$ is bounded between $+\sum |a_k|$ and $-\sum |a_k|$ since $-1 \leq x \leq 1$. Also, $f_n \to f$ uniformly. This is because you can bound the distance between $f_n(x)$ and $f(x)$ by something like $x^n \max a_k/(1-x)$ which is a monotonically decreasing expression in n since $x \in [-1, 1]$. Thus, since the f_n are continuous and bounded and uniformly converges to f, f is also continuous.
For the second part, we can see that $\sum n^{-2} = \pi^2/6 < +\infty$ so we can apply the above theorem.

3 Q 3

We show that f converges on $(-1, 1)$. Consider any $a \in (0, 1)$. We will show that f has uniform convergence on $[-a, a]$ using Weierstrass M test. Define $g_n(x) = x^n$. Then, we have $|g_n(x)| \leq a^n$ for all n. Also, we have $\sum a^n = \frac{a}{1-a}$ which converges since $a < 1$. Therefore, by Weierstrass M test, $\sum g_n(x)$ (which is equal to f converges uniformly on $[-a, a]$, as desired.
Consider $f(x) - f_n(x)$ where $f_n(x) = \sum_{k=1}^{n} x^k$. This is $x^{n+1} + x^{n+2} + \cdots = \frac{x^{n+1}}{1-x}$. Now, consider a $\epsilon > 0$. For any n, we can always find a x such that this difference is greater than ϵ. This is because the difference is an increasing function of x (numerator increases and denominator decreases) and it is unbounded as $x \to 1$. This shows that f does not converge uniformly.