Math 104 Homework 9

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1 Q 1

Consider the textbook function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases}$$

We know that this is infinitely differentiable at 0. We also know it approaches 0 when $x \to 0$. Now consider the function $g(x) = 1 - f(1-x)$. This function approaches 1 when $x \to 1$ and is also infinitely differentiable there. So how do we connect these two? Well what we do is that we scale $f(x)$ by $0.5/f(0.5)$. The good thing with this is that $g(0.5)$ is also 0.5 now. This means they are connected. And since f is infinitely differentiable, so is this new piecewise function. So our answer is something like

$$h(x) = \begin{cases} 0 & x \leq 0 \\ \frac{0.5}{e^{-2}} e^{-1/x} & 0 < x \leq 0.5 \\ \frac{0.5}{e^{-2}} (1 - e^{-1/(1-x)}) & 0.5 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

2 Q 2

We look at the primitive of

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

This is

$$C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 + \cdots + \frac{C_n}{n+1} x^{n+1}$$

Evaluating the primitive at $x = 1$, we find that it is equal to

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$$

which is 0. Evaluating the primitive at $x = 0$, it is just 0. Since it is 0 everywhere, by the mean value theorem there must be some point $x \in [0, 1]$ in which the derivative of the function is 0. However, the derivative is just the function we are looking at. This proves the statement.

3 Q 3

We have that $\frac{f(t)-f(x)}{t-x}$ is the slope of f between the points t and x, which means it is the value of $f'(a)$ for some $a \in (x,t)$ (or (t,x)). Thus, the expression can be replaced with

$$|f'(a) - f'(x)| < \epsilon$$

where $|a - x| < \delta$. Since f' is continuous over a compact set, it is uniformly continuous, so such a δ exists.

4 Q 4

We differentiate it $n-1$ times. We get

$$f^{(n-1)}(t) = tQ^{(n-1)}(t) + (n-1)Q^{(n-2)}(t) - \beta Q^{(n-1)}(t)$$

Plugging it in to the formula, we obtain

$$P(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k$$

$$= f(\alpha) + \sum_{k=1}^{n-1} \frac{(\alpha-\beta)Q^{(k)}(\alpha) + kQ^{(k-1)}(\alpha))}{k!}(\beta-\alpha)^k$$

$$= f(\alpha) + \sum_{k=1}^{n-1} \frac{kQ^{(k-1)}(\alpha)}{k!}(\beta-\alpha)^k - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

$$= f(\alpha) + \sum_{k=0}^{n-2} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1} - \sum_{k=1}^{n-1} \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

$$= f(\alpha) - \frac{Q^{(n-1)}}{(n-1)!}(\beta-\alpha)^n$$

This shows that

$$f(\alpha) = P(\beta) + \frac{Q^{(n-1)}}{(n-1)!}(\beta-\alpha)^n$$

5 Q 5

5.1 Part A

Assume f has two fixed points. Then, take those two points and apply the mean value theorem. This means that $f'(x) = 1$ for some x between those two points, which violates the constraint. Therefore, f has less than two fixed points.

5.2 Part B

If f has a fixed point, this means that

$$t = t + (1 + e^t)^{-1}$$

However, this means that

$$(1 + e^t)^{-1} = 0$$

which is impossible. Thus, f has no fixed points.

5.3 Part C

If there were no fixed points, then we would have $|f'(x)| \geq 1$. This is because f must always be above (or below) the line $y = x$ at all points. This violates the constraint, which shows that f has fixed points.

We want to show that $|f(x_n) - x_n| > |f(x_{n+1}) - x_{n+1}|$. Since $f(x) - x$ is a bounded and monotone sequence, it will converge. So now we show it. We want tho show that

$$|f(x_n) - x_n| > |f(x_n) - f(f(x_n))|$$

Now, if this were not true, then we can use the mean value theorem on the points $(x_n, f(x_n))$ and $(f(x_n), f(f(x_n)))$. This would mean $f'(x) \geq 1$ at some point between x_n and $f(x_n)$, violating the constraint. Therefore the statement is true.

5.4 Part D

It can be visualized by the zigzag path because $x_{n+1} = f(x_n)$, so just by drawing out each point $(x_n, f(x_n))$ yields the desired zigzag path.