# HW 3

Keita Tanabe

## 10.6

a) let $(S_n)$ be a sequence such that

$$|S_{n+1} - S_n| < 2^{-n} \quad \text{for all } n \in N$$

Prove $(S_n)$ is a Cauchy sequence and hence is convergent

Def Cauchy Seq:
$\forall \varepsilon > 0$, $\exists N$ s.t. $m, n > N \rightarrow |S_n - S_m| < \varepsilon$

for $S_n$, $2^{-n}$ is never greater than $2^{-1}$

$n=1$ $m=n+1$ as $n$ increases $2^{-n}$ decreases
therefore, $S_{n+1}$ grows closer to $S_n$
This means that as the $n$ increases, the rate of change of the sequence decreases exponentially while $n$ increases linearly. Therefore since $\lim_{x\to\infty} \frac{x}{a^x} = 0$ $\forall a > 1$, this seq converges.
Therefore, since it converges, it is a cauchy sequence.

b) Is the result in (a) true if we only assume $|S_{n+1} - S_n| < \frac{1}{n}$ $\forall n \in N$

In this case, the result in (a) is not necessarily valid. The case was based on one side decreasing at an exponential rate while the other increased at a linear rate, therefore, as $n$ increases, $|S_m - S_n)$ will increase without limit if $m$ increases and $n$ stays constant.

## 11.2

Consider

$a_n = (-1)^n$ $\qquad b_n = \frac{1}{n}$ $\qquad C_n = n^2$ $\qquad d_n = \frac{6n+4}{7n-3}$

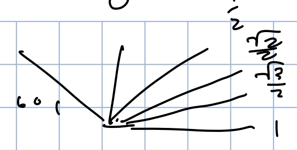
a) a monotone subsequence of $(a_n)$ could be $a_n$ $\forall n \in N_{even}$

$(b_n)$ is a monotone sequence, so any subseq is monotone decreasing

$(C_n)$ is monotone, so any subseq is monotone increasing

$(d_n)$ is monotone, so any subseq is monotone decreasing

c) $\limsup (a_n) = 1$ $\quad \liminf (a_n) = -1$
$\lim (b_n)$ exists $\therefore \lim(b_n) = \limsup(b_n) = \liminf(b_n) = 0$
$\lim (C_n) = +\infty = \limsup(C_n) = \liminf(C_n)$
$\lim (d_n) = \frac{6}{7} = \limsup(d_n) = \liminf(d_n)$




b) $a_n$ : $S = \{1, -1\}$
$b_n$ : $S = \{0\}$
$c_n$ : $S = \{+\infty\}$
$d_n$ : $S = \{\frac{6}{7}\}$

d) $a_n$ does not converge nor diverge, it oscillates between 1 and $-1$, $b_n$ converges to 0, $C_n$ diverges to $+\infty$, $d_n$ converges to $\frac{6}{7}$.

e) $a_n$ is bounded as $|a_n|$ never exceeds 1, $b_n$ is bounded as $|b_n|$ never exceeds 1 $\forall n > 0 \in \mathbb{N}$, $c_n$ is not bounded as it diverges, $d_n$ is bounded as it never exceeds $\frac{5}{2}$.

11.3

$S_n = \cos\left(\frac{\pi n}{3}\right)$ $t_n = \frac{3}{4n+1}$ $u_n = \left(-\frac{1}{2}\right)^n$ $V_n = (-1)^n + \frac{1}{n}$

a) a monotone subseq for $S_n$ is the seq of peaks, which would be $S_n$ for all $n$ that are multiples of 6
$t_n$ is monotone decreasing, therefore any subseq will be monotone decreasing, let the subseq be $t_n$ $\forall$ $n > 100$!
a monotone subseq of $u_n$ could be the seq $U_n$ for all odd $n$, which would be monotone increasing
a monotone subseq of $V_n$ could be $V_n$ where $n$ is only even, which would be monotone decreasing.

b) $s_n$ : $S = \{1, \frac{1}{2}, -\frac{1}{2}, -1,\}$
$t_n$ : $S = \{0\}$
$u_n$ : $S = \{0\}$
$v_n$ : $S = \{1, -1\}$

c) $\limsup(S_n) = 1$ $\liminf(S_n) = -1$
$\limsup(t_n) = \liminf(t_n) = \lim(t_n) = 0$
$\limsup(u_n) = \liminf(u_n) = \lim(t_n) = 0$
$\limsup(V_n) = 1$ $\liminf(t_n) = -1$

d) none of these sequences diverge, and only $t_n$ and $u_n$ converge (to 0)

e) all of these sequences are bounded. and none of their magnitudes ever exceed 2.

11.5 Let $(q_n)$ be an enumeration of all of the rationals in the interval $(0,1]$

Since $(q_n)$ is an enumeration of a set of rational numbers, it must be an oscillating set, starting with 1 and moving closer to 0 as the denominator (of each element) increases, then grows closer to 1, as the numerator grows larger, and then the denominator again, and so on

As the denominator of the rational numbers grows closer to $+\infty$, and the numerator follows suit, an infinite number of elements growing closer to any rational number are possible, therefore meaning that there are subsequences that converge to any rational number in $(0,1]$

Therefore the set of subsequential limits is the set of all rational numbers in $[0,1]$

b) $\liminf(q_n) = 0$ $\limsup(q_n) = 1$

2) Limsup is the natural endpoint for the subsequence of maximums in any oscillating seq, or it is the limit for any sequence for which the limit exists. Limsup is different to just sup because it doesn't care about any of the finite values. Limsup only cares about how a sequence acts when it stretches on forever. The weirdest thing about limsup is probably that when a sequence is unbounded or it becomes incredibly large at some point (i.e $S_n = \frac{1}{n-6}$) but still converges to zero, as sup is large but limsup is zero.

Something that seems correct but is actually wrong is that $\limsup \frac{99}{n} = 99$, or that limsup is always larger than sup because limsup takes into account when n is infinite.