HW 04

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Ross
12.10

Prove (S_n) is bounded iff $\limsup |S_n| < +\infty$

if (S_n) is bounded, then there exists some $\varepsilon > 0$ such that $\forall n > 0$, $\varepsilon > |S_n|$

so, if we can show this ε exists, then $|S_n|$ is bounded

for any finite n, if S_n is defined, $|S_n|$ will be finite, therefore for any finite n, $|S_n| < +\infty$, and the discussion of whether the seq is bounded is put upon n at infinity.

If we take S to be the set of subsequential limits of (S_n), then $\sup S = \limsup S_n$, meaning that all subsequential limits are less than or equal to $\limsup S_n$

S represents all possible outcomes of S_n at infinity, so since $\limsup S_n$ represents $\sup S$, then if $\limsup |S_n| = +\infty$, then it is not bounded, and if $\limsup |S_n| < +\infty$, it is bounded.

Similarly, if we assume that (S_n) is bounded, then we must know that there are no infinite outcomes as $n \to \infty$, therefore $\limsup |S_n| < +\infty$.

12.12 let (S_n) be a sequence of non negative numbers, and for each n define:
$\sigma_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$

a) Show

$$\liminf S_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup S_n$$

$\lim \sigma_n = \frac{1}{\infty}(s_1 + s_2 + \ldots + s_\infty)$

$\liminf \sigma_n = \frac{1}{\infty}(s_1 + s_2 + \ldots + \liminf S_n)$

since we are adding an infinite number of non negative values together,
$s_1 + s_2 + \ldots + \liminf S_n > \liminf S_n$

if $s_1 + s_2 + \ldots + \liminf S_n$ converges to a value, because all elements are non negative, we could say it converges absolutely, meaning that in this case, $\lim S_n = 0$, this would imply $\liminf S_n = 0$, and since any finite value divided by infinity is zero $\liminf \sigma_n = 0 \geq 0$

if (S_n) diverges, then the sum $s_1 + s_2 + \ldots s_n$ diverges as well, and even if it is divided by $\frac{1}{n}$ $n \to \infty$, since there are n elements in the series, and all values are non negative, the sum of all values would still diverge, meaning $\lim S_n = +\infty$, $\liminf S_n = +\infty$, $\liminf \sigma_n = +\infty$

The middle inequality is true by definition.

$\limsup \sigma_n = \frac{1}{n \to \infty}(s_1 + s_2 + s_3 + \ldots + \limsup S_n)$

assume $\limsup \sigma_n > \limsup S_n$

show $\limsup \frac{1}{n}(s_1 + s_2 + \ldots s_n) > \limsup S_n$

The case where $\limsup S_n$ is $+\infty$ is superfulous, as $+\infty$ is greater than or equal to anything.

if S_n is bounded, then for every element in $S_1+S_2+S_3+\dots S_n$, it will be a finite number, divided by ∞, thereby making it $0+0+\dots 0=0$, which will be less than or equal to any convergent limit of (S_n).

b) Show if $\lim S_n$ exists, then $\lim \sigma_n$ exists, and $\lim S_n = \lim \sigma_n$

if S_n exists $\longrightarrow$ $\liminf S_n = \limsup S_n$

since $\liminf S_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup S_n$

$\liminf S_n = \liminf \sigma_n = \limsup \sigma_n = \limsup S_n$

therefore $\lim \sigma_n$ exists and $\lim \sigma_n = \lim S_n$.

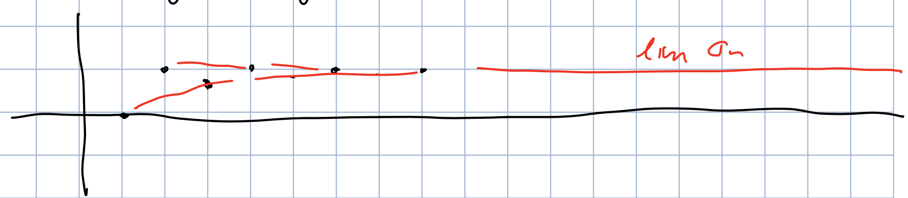
c) Give an example where $\lim \sigma_n$ exists but $\lim S_n$ does not

let $S_n = 1+(-1)^n$

since S_n oscillates between 0 and 2, $\lim S_n$ does not exist.

$\sigma_n = \frac{1}{n}\left(1+(-1)^1 + 1+(-1)^2 + \dots 1+(-1)^n\right)$

Because σ_n sums all previous results and averages them, σ_n will continuously average 2 and 0, oscillating between 1 and values gradually approaching 1 from below, looking something like.



which we can view as two monotone subsequences of only 1s and hyperbolically approaching 1.

Therefore $\lim \sigma_n$ exists.

14.2

a) $\sum \frac{n-1}{n^2} = \sum \frac{n}{n^2} - \frac{1}{n^2} = \sum \frac{1}{n} - \sum \frac{1}{n^2}$, since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges, this is effectively an infinite minus some finite number, meaning that this is infinite, and thus divergent.

b) $\sum (-1)^n$ This constantly oscillates between 1 and 0, and is therefore bounded, since there is never any change in behavior, it does not converge and simply oscillates.

c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$ The series of any constant over a quadratic converges, so this must converge.

d) $\sum \frac{n^3}{3^n} = \sum \left(\frac{n}{\sqrt[3]{3}}\right)^3$ Since an exponential grows faster than any polynomial, after some $n > 0$, $\frac{n^3}{3^n} < \frac{1}{n^2}$, and since it would happen at a finite value, the total would be finite, therefore, this converges

e) $\sum \frac{n^2}{n!}$ $\quad \limsup \dfrac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \limsup \dfrac{\frac{(n+1)^2}{n+1}}{n^2} = \limsup \frac{n+1}{n^2} = 0$

therefore by ratio, this converges.

f) $\sum \frac{1}{n^n}$ $\quad$ let $\alpha = \limsup \left(\frac{1}{n^n}\right)^{\frac{1}{n}} = \limsup \left(n^{-(n)}\right)^{\frac{1}{n}} = \limsup (n^{-1})$
$= 0$, therefore converges by root test.

g) $\sum \frac{n}{2^n}$ $\quad$ let $\alpha = \limsup \left(n2^{-n}\right)^{\frac{1}{n}} = \limsup n^{\frac{1}{n}}(2^{-1}) = \frac{1}{2}\limsup n^{\frac{1}{n}}$
$= \frac{1}{2} < 1$ therefore by the ratio test. This converges.

14.10

Find a series $\sum a_n$ which diverges by Root test but for which ratio test gives no information. Compare example 8

$\limsup (a_n)^{\frac{1}{n}} > 1$, $\quad \limsup \left(\frac{a_{n+1}}{a_n}\right) \geq 1 \geq \liminf \left(\frac{a_{n+1}}{a_n}\right)$

let $a_n = 3^{(-1)^n}(3)^n$

$$\left(3^{(-1)^n}3^n\right)^{\frac{1}{n}} = \left(3^{\left(\frac{(-1)^n+n}{n}\right)}\right) = (3)\left(3^{\frac{(-1)^n}{n}}\right)$$

$\limsup 3\left(3^{\frac{(-1)^n}{n}}\right) = 3(3^0) = 3 > 1$ $\quad \therefore$ diverges by root test

$$\frac{a_{n+1}}{a_n} = \frac{3^{(-1)^{n+1}}(3^{n+1})}{3^{(-1)^n}(3^n)} = \frac{3\left(3^{(-1)^{n+1}}\right)}{3^{-n}}$$

$$\limsup \frac{3\left(3^{(-1)^{n+1}}\right)}{3^{(-1)^n}} = \frac{3(3^1)}{3^{-1}} = 9 \geq 1$$

$$\liminf \frac{3\left(3^{(-1)^{n+1}}\right)}{3^{(-1)^n}} = \frac{3\left(3^{(-1)}\right)}{3^1} = \frac{3}{9} = \frac{1}{3} \leq 1$$

$\therefore$ the ratio test gives no information.

This is extremely similar to that on example 8, but instead of $-n$, it is $+n$, which is the difference between a exponential in the numerator and the denominator.

Rudin

6. a) $a_n = \sqrt{n+1} - \sqrt{n}$

observe:

this is of the form $f(a+1) - f(a)$, which means it telescopes

$$\sum a_n = \cancel{\sqrt{1+1}} - \sqrt{1} + \cancel{\sqrt{2+1}} - \cancel{\sqrt{2}} + \sqrt{3+1} - \cancel{\sqrt{3}} \cdots + \cancel{\sqrt{n}} - \sqrt{n-1} + \sqrt{n+1} - \cancel{\sqrt{n}}$$

$\therefore \sum a_n = \lim \sqrt{n+1} - \sqrt{1}$, $\lim \sqrt{n+1} - 1 = \infty$, so this diverges.

b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{\sqrt{n+1}}{n} - \frac{\sqrt{n}}{n} = \frac{\sqrt{n+1}}{n} - \frac{\sqrt{n}}{n} \frac{(n^{-\frac{1}{2}})}{n^{-\frac{1}{2}}} = \frac{\sqrt{n+1}}{n} - \frac{1}{\sqrt{n}}$

$$\sum a_n = \sum \frac{\sqrt{n+1}}{n} - \frac{1}{\sqrt{n}} = \sum \frac{\sqrt{n}(\sqrt{n+1}) - n}{n^{\frac{3}{2}}}$$

notice $\sqrt{n}(\sqrt{n+1})$ has the same power as n, and thus both grow at a similar rate, therefore, this is similar to having a constant over a $n^{\frac{3}{2}}$ at high n, therefore this converges.

c) $a_n = (\sqrt[n]{n} - 1)^n$

$\limsup ((\sqrt[n]{n} - 1)^n)^{\frac{1}{n}} = \alpha = \limsup \sqrt[n]{n} - 1 = \limsup n^{\frac{1}{n}} - 1 = 0$

therefore by root test this converges

d) $a_n = \frac{1}{1+z^n}$ for complex z

I have no clue how to approach this

7. Prove the convergence of $\sum a_n$ implies the convergence of

$\sum \frac{a_n}{n}$ let $b_n = \frac{a_n}{n}$

if $a_n \geq 0$

Consider $\sum a_n$ converges, therefore, any series c_n such that $\forall i \; c_i \leq a_i$, c_n must converge as well

$b_n = \sum \frac{a_n}{n}$, for any n, $\frac{a_n}{n} \leq a_n$, because $n \geq 1$

therefore $\sum \frac{a_n}{n}$ must converge.