HW 05

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13.3

Let B be the set of all bounded sequences $x = (x_1, x_2, x_3 \dots)$ and define $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\}$

a) show d is a metric for B

① show $d(x,y) \geq 0$ $\quad d(x,y) = 0 \iff x = y$

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \dots\}$

if $\sup\{|x_j - y_j| : j = 1, 2, 3 \dots\} = 0$, then $\forall_j$ $x_j = y_j$, as since it is measuring the absolute value, there cannot be any negative values, and any absolute difference would cause a non zero sup.

② show $d(x,y) = d(y,x)$

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2 \dots\}$

$= \sup\{|-(y_j - x_j)| : j = 1, 2 \dots\}$

$= \sup\{|y_j - x_j| : j = 1, 2 \dots\} = d(y,x)$

③ $d(x,y) + d(y,z) \geq d(x,z)$

$\sup\{|x_j - y_j| : j = 1, 2, 3 \dots\} + \sup\{|y_i - z_i| : i = 1, 2, 3 \dots\} \geq \sup\{|x_k - z_k| : k = 1, 2, \dots\}$

The sup is larger than or equal to the max of the set therefore the sup is larger than or equal to the largest difference between each seq.

The largest difference between x and z can only approach the addition of that of x and y and y and z if x and z are further than y to either.

this would mean that y would be in the middle of the two, in which case the expressions would be equal, and the inequality holds.

all conditions hold, therefore this is a metric

b) does $d^* = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric of B?

Yes, since $|x_j - y_j|$ satisfies all three axioms for any j the summation of all $|x_j - y_j|$ would also hold.

13.5 a) verify $\bigcap\{S\setminus U : U\in\mathcal{U}\} = S\setminus\bigcup\{U : U\in\mathcal{U}\}$

let $x\in\bigcap\{S\setminus U : U\in\mathcal{U}\}$ $\rightarrow$

therefore $x\in S\setminus U$ $\forall U\in\{U\}$

$x\notin U$ $\forall U\in\mathcal{U}$

meaning $x\in S\setminus\bigcup\{U : U\in\mathcal{U}\}$

let $x\in S\setminus\bigcup\{U : U\in\mathcal{U}\}$ $\leftarrow$

$x\notin U$ $\forall U\in\mathcal{U}$

and $x\in S\setminus U$ $\forall U\in\mathcal{U}$

therefore $x\in\bigcap\{S\setminus U : U\in\mathcal{U}\}$

b) show the intersection of closed sets is a closed set

take $\mathcal{a}$ to be the set of closed sets in S

by def. $S\setminus E$ is open $\forall E\in\mathcal{a}$

take $\mathcal{U}=\{S\setminus E : E\in\mathcal{a}\}$

reciprocate to $\mathcal{a}=\{S\setminus u : u\in\mathcal{U}\}$ given $\forall E\in\mathcal{a}$ $E=S\setminus(S\setminus E)$

then

$\bigcap\{E\in\mathcal{a}\}=\bigcap\{S\setminus U : U\in\mathcal{U}\} = S\setminus\bigcup\{U : U\in\mathcal{U}\}$

by de Morgan's law

we take $S\setminus$ of both to get

$S\setminus\bigcap\{E\in\mathcal{a}\}=\bigcup\{U : U\in\mathcal{U}\}$

since the union of open sets is open, both must be open, meaning that $S\setminus\{E\in\mathcal{a}\}$ is open, meaning the intersection of closed sets is closed.

13.7 imagine some $x\in X$ for open set X

then exists some open interval Y such that $x\in Y$ and $Y\subset X$

there is one largest Y for any X, and if we take the set of all largest Ys, then we may span all of X, thereby spanning all of X.

4. Prove that taking closure on a closed set does not make the set bigger.

if we take a closed subset to be one where it has all ends of sequences in an open set. By taking the limits of subseq. in the open set and including them you get a closed set.
Therefore this set is already closed and all subseq already have their limits included, meaning taking closure again does nothing.

5. Prove $\bar{S}$ is the intersection of all closed subsets in X that contain S.

Given a non zero space X, there are an infinite number of closed subsets of X containing S, including $\bar{S}$. All of these sets also contain $\bar{S}$, meaning that the intersection of all of them is at least as small as $\bar{S}$
By definition, $\bar{S}$ is the smallest closed set containing S, therefore, there is no closed set that does not contain $\bar{S}$, and $\bar{S}$ is in these sets meaning the intersection is $\bar{S}$.