HW 6 Keita Tanabe

1) Can you prove $[0,1]^2$ in $\mathbb{R}^2$ is compact given that $[0,1]$ is sequentially compact?

$[0,1]^2$ is $[0,1] \times [0,1]$

Both are sequentially compact, so if two sequentially compact spaces X, Y $\to X \times Y$ is sequentially compact, then $[0,1]^2$ will be shown to be sequentially compact.

X is sequentially compact, therefore every sequence of elements in X has a convergent subsequence
Same for Y
if: we imagine $X \times Y$ in chunks of sequences of X, then similarly, the elements of $X \times Y$ will be situated such that there are sequences of X and Y.

Since there are an infinite number of convergent elements in a convergent subsequence, and there is a convergent subsequence in both the X and Y, there must be infinite overlap between these subsequences, From these overlapping values, we can make a subsequence where both elements of X and Y are converging, therefore in the sequence, there is a convergent subsequence. If we extend this to any and all sequences, we find that all sequences in $X \times Y$ have a convergent subseq.

Therefore if X and Y are subsequentially compact, then $X \times Y$ is as well.
Therefore since $[0,1]$ is subsequentially compact, $[0,1]^2$ is as well.

2) Let E be the set of points $x \in [0,1]$, whose decimal expansion consist of only 4 and 7
Is E countable? is E compact?

If we imagine that E is countable and we list the decimals as such

1. 0.4444…
2. 0.7777…
3. 0.4747…
4. 0.7474…

assume E is countable and we have listed all possible combinations
if we make a new number by taking the diagonal and reversing ($4 \to 7$, $7 \to 4$) the final digit, we can create a new number in the set not in the listing, therefore by contradiction, E is not countable.

All items in the set are bounded, as $\nexists\, x > 1$
Consider the upper limit of the set. Can we name the sup of the set? is it within the set?
The largest element is the infinitely long decimal $0.7777\ldots777$
Therefore we can write it as a closed interval between
$[0.4, 0.777\ldots77\ldots]$
Therefore it is compact as it is closed and bounded in $\mathbb{R}$

3) Let $A_1, A_2 \ldots$ be a subset of a metric space. If $B = \bigcup_i A_i$, then $\overline{B} \supset \bigcup_i \overline{A_i}$
Is it possible that this inclusion is a strict inclusion?

If we consider $A_1, A_2 \ldots$, it can be seen as a finite or countably infinite. In this case, if we close a countably infinite set, then we must add values, think of $\overline{\mathbb{Q}}$, this is $\mathbb{R}$, since $\mathbb{Q}$ is not closed itself. Thinking this way, just like the irrational numbers, there are elements in $\overline{B}$ that are not in $\bigcup_i \overline{A_i}$, therefore it is a strict inclusion.

4) Last time, we showed any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals
"any closed subset of $\mathbb{R}$ is a countable union of closed intervals, Because every closed set is the complement of an open set, and adjacent open intervals sandwitch a closed interval"
Where is the argument wrong? give an example.

The argument is wrong because it could be ~~the~~ uncountable union of closed intervals as well.

An example of this is the set of problem 2, as it is uncountable, and since only digit of 4 and 7 are allowed, no two adjacent numbers are in the set, therefore the set from problem 2 is closed and the union of an uncountable number of closed intervals (each with 1 element).