HW 7

Keita Tanabe

1. If X and Y are open cover compact, can you prove that $X \times Y$ is open cover compact?

If X and Y are open cover compact, then each can be contained in a finite union of open sets

Therefore for every element in X, there is an open set G_i s.t. the element is contained. This also applies to Y

Since there are only a finite number of these sets, either X has to be finite, or there are an infinite amount of elements in some of the open sets.

If we imagine each open subset in X being paired with each open subset corresponding to Y, then if we say that there is a union of n open sets that strictly contains X, and one of m open sets that strictly contains Y, where n and m are finite, then we would have $n * m$ open sets.

Since all items in Y would be contained in the open sets, as would with X, if we imagine some sets of open sets that contain each. The cartesian product of these sets would be a finite set of pairs of open sets. If we then take the cartesian product of the pairs made with the previous product, then, we have a set of open sets that contain all elements of $X \times Y$. This set is finite, as it is the union of $n * m$ sets, where n and m are finite.

These sets are open because the cartesian product does not affect the elements in either group, just pairing them.

Therefore since there is a finite union of open sets that strictly contain all elements of $X \times Y$, $X \times Y$ is open cover compact.

2. Let $f: X \to Y$ be a continuous map between metric spaces. Let $A \subset X$ be a subset decide if the following are true.

- If A is open, then $f(A)$ is open.
  False, if we imagine f is a function that sends all values to 1 (in $\mathbb{R}$) and A is $(0,1)$, then $f(A) = [1]$, which is closed even though A is open

- If A is closed, then $f(A)$ is closed
  False, this is only true if A is compact. A counter-example could be $A = \mathbb{R}$ $f(x) = \frac{1}{x}$, as $\mathbb{R}$ is closed (as well as open) and $f(A) = (0, \infty)$ which is an open set

- If A is bounded, then A is bounded
  True: Since for every conv. sequence in A, there exists a convergent sequence in $f(A)$, and since A is bounded, all sequences are convergent and under a value, intuitively this carries over when there are similar convergent sequences in $f(A)$ (because it is contin.)

- If A is compact, then $f(A)$ is compact.

True: If a set is compact, then it is also sequentially compact, meaning that every sequence has a convergent subsequence. From this, if we take a sequence in $f(A)$ to be f(the sequence in A), since continuous maps preserve convergent sequences, All sequences in $f(A)$ have convergent subsequences, and are therefore compact.

- If A is connected then $f(A)$ is connected.

True: If a set is connected, then it cannot be written as a disjoint union of two non empty open sets. In this case, for any x, y in A, $[x,y] \subset A$ since f is continuous, $[f(x), f(y)]$ will be connected if $[x,y]$ is connected, as it will apply to every element.

3. Prove there is no surjective map from $[0,1] \to \mathbb{R}$, s.t. $\mathbb{R}$ is surjective.

Assume there exists a surjective map from $[0,1] \to \mathbb{R}$
then the map, let it be named $f\ [0,1] \to \mathbb{R}$, must be continuous, such that no element of $\mathbb{R}$ is left out.
Thus, we say that $\mathbb{R} = f([0,1])$ / however, $[0,1]$ is bounded, and $\mathbb{R}$ is not, therefore, by the previous discussion, this is a contradiction, as the continuous map of a bounded set must be bounded.

Therefore there does not exist a surjective map from $[0,1]$ to $\mathbb{R}$.