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1.10 Prove $(2n+1)+(2n+3)+(2n+5)+\dots+(4n-1)=3n^2$ for all positive integers n.

let $\mathbb{Z}_{odd} = \{1, 3, 5, 7, \dots\}$ $\quad \mathbb{Z}_{odd_1} = 1$, $\mathbb{Z}_{odd_2} = 3 \dots$

$$(2n+\mathbb{Z}_{odd\,m}) = (4n-1)$$
$$2n = \mathbb{Z}_{odd\,m}$$
$$\mathbb{Z}_{odd\,m} = 2n-1$$

$\therefore$ for $n=1$, $2n+\mathbb{Z}_{odd\,m} = 4n-1$ when $\mathbb{Z}_{odd\,m} = 1$

for $n=2$, $2n+\mathbb{Z}_{odd\,m} = 4n-1$ when $\mathbb{Z}_{odd\,m} = 3 \dots$

$n=3$, $2n+\mathbb{Z}_{odd\,m} = 4n-1$ when $\mathbb{Z}_{odd\,m} = 5$

each integer adds its number of $2n+\mathbb{Z}_{odd}$ terms, meaning for any int n, the left hand side becomes

$$\underbrace{(2n+1)+\dots\dots}_{n \text{ terms}} = 3n^2$$
$$\downarrow$$
$$2n^2 + \underbrace{1+3+\dots}_{n \text{ odd int}} = 3n^2$$

Show that the sum of the first n odd integers is equal to n^2

Since the highest odd integer for any given n is $2n-1$ we can express the sum of the odd ints as

$$\underbrace{1+\dots(2n-1)}_{n \text{ terms}}$$

$$\therefore \quad = \underbrace{n(1)+(n-1)(2)+(n-2)(2)+(n-3)(2)}_{n \text{ terms}} \dots$$

$$= n^2 + (n-1)(n) - 2(\underbrace{1+2+\dots}_{n-1})$$

Sum of integers formula:

$$S = \frac{n_{int}(\text{first}+\text{last})}{2}$$

$n_{int} = n-1$
first $= 1$
last $= n-1$

$\therefore$

$$n^2+(n-1)(n) - 2\left(\tfrac{1}{2}(n-1+1)\right)$$
$$= n^2$$

$\therefore$ the n odd ints $= n^2$

$\therefore \quad 2n^2 + \underbrace{1 + 3 + \cdots}_{n \text{ odd int}} = 3n^2$

$$2n^2 + n^2 = 3n^2$$
$$3n^2 = 3n^2$$

$\therefore \ (2n+1) + (2n+3) + \ldots (4n-1) = 3n^2 \ \forall \ n \in \mathbb{Z} \ni n > 0$

1.12 for $n \in \mathbb{N}$, let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdot \ldots n$
Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for } k = 0, 1, \ldots n$$

The binomial theorem asserts that

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots \binom{n}{n} b^n$$

$$= a^n + n a^{n-1} b + \tfrac{1}{2} n(n-1) a^{n-2} b^2 + \ldots + n a b^{n-1} + b^n$$

a) let $n = 1$

$$(a+b)^1 = \binom{1}{0} a^1 + \binom{1}{1} b^1 = \frac{1!}{0!(1-0)!} a^1 + \frac{1!}{1!(1-1)!} b^1 = a+b \ \checkmark$$

let $n = 2$

$$(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} b^2$$

RHS $= \frac{2!}{0!(2-0)!} a^2 + \frac{2!}{1!(2-1)!} ab + \frac{2!}{2!(2-2)!} b^2 = \frac{2!}{2!} a^2 + \frac{2}{1} ab + \frac{2!}{2!} b^2$

$= a^2 + 2ab + b^2$

LHS $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

LHS = RHS $\checkmark$

let $n = 3$

$$(a+b)^3 = \binom{3}{0} a^3 + \binom{3}{1} a^2 b + \binom{3}{2} ab^2 + \binom{3}{3} b^3$$

RHS $= \frac{3!}{0!(3-0)!} a^3 + \frac{3!}{1!(2)!} a^2 b + \frac{3!}{2!(1)!} ab^2 + \frac{3!}{3!(0)!} b^3$

$= a^3 + \frac{6}{2} a^2 b + \frac{6}{2} ab^2 + b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

LHS $= (a+b)(a+b)(a+b) = (a^2 + 2ab + b^2)(a+b) = a^3 + a^2 b + 2a^2 b + 2ab^2 + b^2 a + b^3$

$= a^3 + 3a^2 b + 3ab^2 + b^3 =$ RHS $\checkmark$

b) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

LHS $\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!\,(k!(n-k)! + (k-1)!(n-k+1)!)}{k!(n-k)!(k-1)!(n-k+1)!}$

$\frac{n!(n-k)!(k! + (k-1)!(n-k+1))}{k!(n-k)!(k-1)!(n-k+1)!} = \frac{n!(k! + (k-1)!(n-k+1))}{k!(k-1)!(n-k+1)!}$

$$\frac{n!\,(k! + (k-1)!\,(n-k+1))}{k!\,(k-1)!\,(n-k+1)!} = \frac{n!\,(k-1)!\,(k+n-k+1)}{k!\,(k-1)!\,(n-k+1)!}$$

$$= \frac{n!\,(n+1)}{k!\,(n-k+1)!} = \frac{(n+1)!}{k!\,(n+1-k)!}$$

RHS $\binom{n+1}{k} = \frac{(n+1)!}{k!\,(n+1-k)!}$

RHS = LHS ✓

c) $(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \dots \binom{n}{n} b^n$

from a) $n=1$ verified

verify for $n+1$

$(a+b)^{n+1} = \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \dots + \binom{n+1}{n} a b^n + \binom{n+1}{n+1} b^{n+1}$

$= (a+b)^n (a+b)$

2.1 Show $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{81}$ are not rational numbers.

$\sqrt{3}$, assume $\sqrt{3}$ is a rational number and equals $\frac{n}{k}$ ∋ $n, k \in \mathbb{Z}$ and are coprime

$\sqrt{3} = \frac{n}{k}$ $\quad 3 = \frac{n^2}{k^2}$ $\quad n^2$ is odd or k^2 is odd, and the other is even

assume n is even, k is odd

let $n = 2\alpha$, $k = 2\beta + 1$

$n^2 = 4\alpha^2$, $k^2 = 4\beta^2 + 2\beta + 1$

$n^2 = 2(2\alpha^2)$ even $\quad k^2 = 2(2\beta^2 + \beta) + 1$ odd

$3k^2 = n^2$, (odd) odd = even ✗

an odd int multiplied with an odd int is always odd

∴ $\sqrt{3}$ is not rational, if n is even, k is odd

assume n is odd, k is even.

$n = 2\beta + 1$ $\qquad k = 2\alpha$

$n^2 = 2(2\beta^2 + \beta) + 1$ $\qquad k^2 = 2(2\alpha^2)$

$3 = \frac{n^2}{k^2}$ $\qquad 3k^2 = n^2$ $\qquad \overset{\text{even}}{2(3(2\alpha^2))} = \overset{\text{odd}}{2(2\beta^2 + \beta) + 1}$ ✗

∴ $\sqrt{3}$ is not rational if n is odd, k is even

∴ in both cases $\sqrt{3}$ is not rational ∴ $\sqrt{3}$ is not rational

$\sqrt{5}$, assume $\sqrt{5}$ is a rational number and equals $\frac{a}{b}$ ∋ $a, b \in \mathbb{Z}$ & a, b coprime

$\sqrt{5} = \frac{a}{b}$ $\qquad 5 = \frac{a^2}{b^2}$ $\qquad b^2 = \frac{a^2}{5}$, since b^2 is the product of two integers, it is also an integer. ∴ $\frac{a^2}{5}$ is an integer

∴ a^2 is a multiple of 5 $\quad$ let $\gamma = \frac{a^2}{5}$

if $b^2 = \gamma$, then b^2 and a^2 are not coprime

∴ a and b are not coprime

∴ $\sqrt{5}$ is not rational by contradiction

$\sqrt{7}$, assume $\sqrt{7}$ is a rational number equal to $\frac{n}{k}$ ∋ $n, k \in \mathbb{Z}$ and n, k coprime

$\sqrt{7} = \frac{n}{k}$

$7 = \frac{n^2}{k^2}$ $\qquad 7k^2 = n^2$

since n^2 & k^2 are squared, they must have an even number of prime factors (2(factors of $n \oplus k$))

However 7 is a prime number

∴ n^2 has an even and an odd number of prime factors (even due to being squared, odd from a square number +1 factor)

✗ ∴ $\sqrt{7}$ cannot be a rational number

$\sqrt{24}$ $\qquad \sqrt{24} = \sqrt{6 \cdot 4} = 2\sqrt{2}(\sqrt{3})$, $\sqrt{24}$ is even.

let $\sqrt{24}$ be a rational number equal to the quotient of a coprime int n, k

∋ n is even and k is odd

$2\sqrt{2}\sqrt{3} = \frac{n}{k}$

$2\sqrt{6}\,k = n$ let $n = 2\alpha$

$\sqrt{6}\,k = \alpha$

$6k^2 = \alpha^2$

α^2 is a multiple of 6, and since α must be an integer due to being half of an even number n, either $k^2 = 6$, which is a contradiction to the original statement that $n \in \mathbb{Z}$, or a contradiction to the statement that n, k are coprime
$\therefore \sqrt{24}$ cannot be rational.

$\sqrt{31}$ 31 is a prime number, therefore the argument for $\sqrt{7}$ also applies to $\sqrt{31}$.

2.2 Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, and $\sqrt[4]{13}$ are not rational numbers

assume $\sqrt[3]{2}$ is a rational number $\ni \sqrt[3]{2} = \frac{n}{k}$ $n, k \in \mathbb{Z}$ & coprime

$\therefore\ 2 = \frac{n^3}{k^3}$ $\quad 2k^3 = n^3$

if n^3 is $2k^3$, then k, n cannot be coprime, as through prime factorization, k^3 and n^3 must both have a number of factors that is a multiple of 3, however, according to $2k^3 = n^3$, n^3 has a number of factors equal to $3a$ and $3b+1$ simultaneously (where $a, b \in \mathbb{N}$), which is impossible

$\therefore\ 2k^3 \neq n^3$ is a contradiction

$\therefore\ \sqrt[3]{2}$ is not rational.

$\sqrt[7]{5}$, since 5 is a prime number, the argument for $\sqrt[3]{2}$ will translate to prove $\sqrt[7]{5}$ is not rational.

$\sqrt[4]{13}$, 13 is a prime number, $\therefore$ the argument is the same as was for $\sqrt[3]{2}$.

2.7 Show the following are rational.

a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{3+1+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{\sqrt{3}^2 + 2\sqrt{3} + 1^2} - \sqrt{3}$

$= \sqrt{(\sqrt{3}+1)^2} - \sqrt{3}$

$= 1 + \sqrt{3} - \sqrt{3} = 1$

1 is a rational number

b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

$= \sqrt{4+2+4\sqrt{2}} - \sqrt{2}$

$= \sqrt{2^2 + 4\sqrt{2} + \sqrt{2}^2} - \sqrt{2}$

$= \sqrt{(2+\sqrt{2})^2} - \sqrt{2}$

$= 2 + \sqrt{2} - \sqrt{2}$

$= 2$

2 is a rational number.

3.6 a) Prove $|a+b+c| \le |a|+|b|+|c| \ \forall a,b,c \in \mathbb{R}$

let $\alpha = a+b$

$|\alpha + c| \le |\alpha| + |c|$ Triangle inequality

$|a+b+c| \le |a+b| + |c|$

$|a+b| \le |a|+|b|$ Triangle inequality

$\therefore$

$|a+b|+|c| \le |a|+|b|+|c|$

$|a+b+c| \le |a+b| + |c| \wedge |a+b|+|c| \le |a|+|b|+|c| \longrightarrow |a+b+c| \le |a|+|b|+|c|$

b) assume $|a_1+a_2+a_3| \le |a_1|+|a_2|+|a_3|$ true

show $|a_1+a_2+a_3+a_4| \le |a_1|+|a_2|+|a_3|+|a_4|$

let $b = a_1+a_2+a_3$

$|b+a_4| \le |b| + |a_4|$ Triangle inequality

$\therefore |a_1+a_2+a_3+a_4| \le |a_1+a_2+a_3| + |a_4|$

from assumption.

$|a_1+a_2+a_3|+|a_4| \le |a_1|+|a_2|+|a_3|+|a_4|$

$\therefore |a_1+a_2+a_3+a_4| \le |a_1|+|a_2|+|a_3|+|a_4|$

$\therefore$ by mathematical induction

$|a_1+a_2+a_3+ \dots a_n| \le |a_1|+|a_2|+|a_3|+ \dots |a_n|$

4.11 consider $a,b \in \mathbb{R}$ where $a<b$, use Denseness of Q 4.7 to show there are infinitely many rationals between a and b

4.7 if $a,b \in \mathbb{R}$ and $a<b$, $\exists \ r \in \mathbb{Q} \ni a<r<b$

for $a, b \in \mathbb{R}$, $\exists \ r_1 \in \mathbb{Q} \ni a<r_1<b$

$\mathbb{Q} \subset \mathbb{R}$

$\therefore$ for a, r_1 $\exists \ r_2 \in \mathbb{Q} \ni a<r_2<r_1$

by mathematical induction, there are infinitely many $r \in \mathbb{Q}$ between a, b.

4.14 Let A and B be nonempty bounded subsets of $\mathbb{R}$.
A+B be the set of all a+b where $a \in A$ & $b \in B$

a) prove $\sup(A+B) = \sup A + \sup B$

$\sup(A+B)$ is an upperbound of A+B
let $\sup(A)$ be γ_1, $\sup(A+B)$ be γ_3
$\sup(B)$ be γ_2

$\gamma_3 > \gamma_1$ $\gamma_3 > \gamma_2$

imagine some element of A+B, a+b
$\ni$ $a = \sup A = \gamma_1$
$\forall\, b \in B$, $\gamma_1 + b \geq \gamma_1$
$\therefore$ if $a = \gamma_1$, any $\gamma_1 + b$ is an upper bound of A
Imagine Sup A+B, which is greater than every other element in A+B
$\therefore$ since $\sup A+B \geq \gamma_1 + b \quad \forall\, b \in B$,
$\sup A+B) - b \geq \gamma_1$
or any $b \in B$ subtracted from sup A+B still results in an upper bound for A

Again, with $\gamma_2 = \sup B$, (any $\gamma_2 + a \quad \forall\, a \in A) \in A+B$ and is an upper bound of B, $\sup A+B \geq \gamma_2 + a$
$\therefore \quad \sup(A+B) - a \geq \gamma_2 \quad \forall\, a \in A$

given ① $\sup(A+B) - a \geq \gamma_2 \quad \forall\, a \in A$, and
② $\sup(A+B) - b \geq \gamma_1 \quad \forall\, b \in B$

let a in ① $= \sup A = \gamma_1$

$\sup(A+B) - \gamma_1 \geq \gamma_2$

$\sup A+B \geq \gamma_1 + \gamma_2 = \sup A + \sup B$

$\nexists\, a \in A \ni a > \sup A$, $\nexists\, b \in B \ni b > \sup B$
$\therefore \quad \sup A+B = \sup A + \sup B$

b) A+B has elements a+b when $a \in A$ and $b \in B$

$\inf(A+B) = a+b$ for some $a \in A$, $b \in B$
and is larger than any other lower bound

inf B and inf(A) are lower bounds of A+B, however, there is the possibility that an a ~~both~~ is outside the set, since A+B is defined by a+b, inf A+B must be the smallest value created by adding the two smallest values $\therefore$ $\inf A+B = \inf A + \inf B$

7.5 a) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$

$$s_n = \sqrt{n^2+1} - n \left(\frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n} \right)$$

$$= \frac{n^2 + 1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n}$$

$$\lim s_n = \lim \frac{1}{\sqrt{n^2+1} + n} = \frac{1}{\infty} = \boxed{0}$$

b) $\lim \left(\sqrt{n^2+n} - n \right)$

$$= \lim \left(n\sqrt{1+\tfrac{1}{n}} - n \right) = \lim n \left(\sqrt{1+\tfrac{1}{n}} - 1 \right)$$

$$\lim \frac{\sqrt{1+\frac{1}{n}} - 1}{\frac{1}{n}} = \lim \frac{-n^{-2}\frac{1}{2}\left(1+\frac{1}{n}\right)^{-\frac{1}{2}}}{-n^{-2}}$$

$$= \lim \frac{1}{2\sqrt{1+\frac{1}{n}}} = \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}}$$

c) $\lim \left(\sqrt{4n^2+n} - 2n \right) = \lim \left(\frac{\sqrt{4n^2+n} - 2n \left(\sqrt{4n^2+n} + 2n\right)}{\sqrt{4n^2+n} + 2n} \right)$

$$\lim \left(\frac{4n^2 + n - 4n^2}{\sqrt{4n^2+n} + 2n} \right) = \lim \frac{n}{n\left(\sqrt{4+\frac{1}{n}} + 2\right)} = \lim \frac{1}{\sqrt{4+\frac{1}{n}} + 2}$$

$$= \frac{1}{\sqrt{4} + 2} = \boxed{\frac{1}{4}}$$