Math 104 Hw 2

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9.9 Suppose there exists N_0 such that $s_n \le t_n$ for all $n > N_0$

a) prove that if $\lim s_n = +\infty$ then $\lim t_n = +\infty$

given that there exists some N_0 s.t. $s_n \le t_n$ $\forall n > N_0$,
then as n goes to infinity, it must pass N_0 at some point.
If N_0 was greater than infinity, then it does not exist
therefore if we take the limit as n goes to infinity, then
n must be greater than N_0.
Therefore $t_n \ge s_n$, thus since $\lim s_n = +\infty$, $\lim t_n$ must also equal $+\infty$.

b) prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$

If N_0 exists such that for every $n > N_0$, $t_n \ge s_n$,
Then understanding that taking the limit, the tail
will exceed N_0, making $s_n \le t_n$.
Therefore if $\lim t_n = -\infty$, and $\lim s_n \le \lim t_n$, then
$\lim s_n = -\infty$.

c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \le \lim t_n$

if $\lim s_n$ exists and $\lim t_n$ exist,
then for every $\varepsilon > 0$, $\exists M > 0$ s.t. if $n > M$ $|s_n - L| < \varepsilon$
and the same for t_n

The tail end of all numbers greater than M and all
numbers greater than N_0 must be the same, therefore,
as we take the limit, it will be as if we take
$n > N_0$, meaning that $s_n \le t_n$ as we take lim
$\therefore \lim s_n \le \lim t_n$

9.15 Show $\lim_{n \to \infty} \frac{a^n}{n!} = 0$ $\forall a \in \mathbb{R}$

$\forall a \in \mathbb{R}$ a can be arbitrarily large, but cannot be infinite

Once n passes a, we can split up the term into two terms.

$$\lim_{n \to \infty} \frac{a^a}{a!} \cdot \frac{a^{n-a}}{n!/a!} \longrightarrow \frac{a^a}{a!} \cdot \frac{a^{n-a}}{(a+1)(a+2)(\ldots)}$$

the first $\frac{a^a}{a!}$ is a real number not equal to zero, so if the
$\frac{a^{n-a}}{n!/a!}$ is 0 or ∞, $\frac{a^a}{a!}$ has no effect on the lim on infinity
as k grows larger from a, a^{n-a} grows by a factor of a, and
$n!/a!$ grows by a factor of $(a+m)$ where $m > 0$

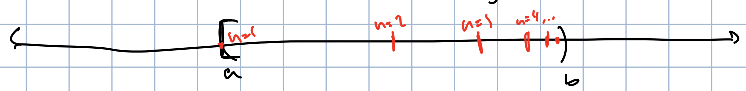
Therefore, since the denominator of the limit grows faster as $n \to \infty$. the lim goes to zero, meaning the entire lim goes to 0, as a real number times 0 is 0.

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S \notin S$. Prove there is a sequence of points in S (s_n) such that $\lim s_n = \sup S$

if S is a nonempty bounded set s.t. $\sup S \notin S$, then S must be bound in (a, b) or $[a, b)$ fashion, where $a, b \in \mathbb{R}$, and $b = \sup S$ and $a \neq \sup S$

Therefore, by the density of rational numbers, between a and b, $\exists$ infinitely many rational numbers, thereby meaning S contains infinitely many numbers.

Considering infinitely many elements in S, it will always be possible to create a sequence by bisecting the set for every n.



Since there are infinitely many rational numbers in S, by letting $s_n = b - \frac{a}{2^n}$, the sequence would stay within S while never reaching $\sup S$. $\lim s_n$ also $= \sup S$.

10.8 Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$ Prove σ_n is an increasing seq.

notice in σ_n, the $(s_1 + s_2 + \dots s_n)$ has n elements all larger than or equal to s_1, therefore let each element be equal to $t_n + s_n$ s.t. $s_1 = t_1 + s_1$, $s_2 = t_2 + s_1$, $s_3 = t_3 + s_1$ ...
we can rearrange so that

$$\sigma_n = \frac{1}{n}(n(s_1) + t_1 + t_2 + t_3 + t_4 \dots + t_n)$$
$$t_1 = 0$$
$$\therefore \sigma_n = \frac{1}{n}(n(s_1) + t_2 + t_3 + \dots t_n)$$

Since (s_n) is an increasing sequence, all $t > 0$ and $t_i < t_{i+1}$

Imagine $\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \dots s_n + s_{n+1})$
$$= \frac{1}{n+1}((n+1)s_1 + t_2 + t_3 + \dots t_n + t_{n+1})$$

if σ_{n+1} is greater than σ_n, then by mathematical induction, (σ_n) is an increasing set

$$\sigma_{n+1} - \sigma_n = \frac{1}{n+1}((n+1)s_1 + t_2 + t_3 + \dots + t_{n+1}) - \frac{1}{n}((n)s_1 + t_2 + \dots + t_n)$$
$$= \frac{1}{n+1}(t_2 + t_3 + \dots + t_n + t_{n+1}) - \frac{1}{n}(t_2 + t_3 + \dots t_n)$$

let $q_1, q_2, \dots q_n, q_{n+1}$ be $q_2 = t_2 - t_2$, $q_3 = t_3 - t_2$
$q_4 = t_4 - t_2$ ...

all q are positive because the series of t are increasing and t_2 is the smallest element

$$= \frac{1}{n+1}\left((n)t_2 + q_3 + q_4 + \dots q_n + q_{n+1}\right) - \frac{1}{n}\left((n-1)t_2 + q_3 + q_{4} + \dots q_n\right)$$

notice $\frac{n}{n+1}t_2 > \frac{n-1}{n}t_2$

if we repeat the last step now letting

$$r_3 = q_3 - q_3 \qquad r_4 = q_4 - q_3 \qquad \dots$$

we get $\left(\frac{n}{n+1} - \frac{n-1}{n}\right)t_2 + \frac{1}{n+1}\left((n-1)q_3 + r_4 + r_5 + \dots r_n + r_{n+1}\right)$
$- \frac{1}{n}\left((n-2)q_3 + r_4 + \dots r_n\right)$

and again $\frac{n-1}{n+1}q_3 > \frac{n-2}{n}q_3$

so if we continue this filling, we find that

$$\sigma_{n+1} - \sigma_n > 0 \quad \therefore \sigma_{n+1} > \sigma_n$$

thus, (σ_n) is an increasing seq

10.9 let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right)s_n^2$ for $n \geq 1$

a) $S_1 = 1 \quad S_2 = \frac{1}{2} \quad S_3 = \frac{1}{6} \quad S_4 = \frac{1}{48} \quad \dots$

b) show $\lim S_n$ exists.

$S_{n+1} = \left(\frac{n}{n+1}\right)S_n^2$ let $n = k = 3, 4, 5 \dots$

therefore

$$S_k = \left(\frac{k-1}{k}\right)S_{k-1}^2$$

$$S_{k-1} = \left(\frac{k-2}{k-1}\right)S_{k-2}^2$$

$$S_{k+1} = \left(\frac{k}{k+1}\right)S_k^2$$

$$S_{k+1} = \left(\frac{k}{k+1}\right)\left(\left(\frac{k-1}{k}\right)S_{k-1}^2\right)^2 = \left(\frac{k}{k+1}\right)\left(\left(\frac{k-1}{k}\right)\left(\left(\frac{k-2}{k-1}\right)S_{k-2}^2\right)^2\right)^2$$

therefore it follows that going from $S_1 = 1$

$$S_k = \left(\frac{k-1}{k}\right)\left(\left(\frac{k-2}{k-1}\right)\left(\dots\dots\left(\left(\left(\frac{1}{2}\right)1^2\right)^2\dots\right)^2\right)^2\right)^2$$

all numerators except the one in S_2 are cancelled out and the denominator grows by a factor of $n-1$ for every S_n, therefore the sequence approaches but never reaches 0, never changing trajectory. therefore $\lim S_n$ exists

c) from b)

$$S_k = \left(\frac{k-1}{k}\right)\left(\left(\frac{k-2}{k-1}\right)\left(\cdots \left(\left(\tfrac{1}{2}\right) 1^2\right)^2 \cdots \right)^2\right)^2$$

$$= \frac{(k-2)^2}{(k)(k-1)} \left(\cdots\cdots \left(\left(\tfrac{1}{2}\right) 1^2\right)^2 \cdots \right)^2$$

$\vdots$

$$= \frac{1^2}{(k)(k-1)(k-2)\cdots(2)} (1)^4$$

$$= \frac{1}{k!}$$

as $k \to \infty$, $\frac{1}{k!}$ goes to 0 if $n = k$, $S_n = \frac{1}{n!}$

$\therefore \lim S_n = 0$

10.10 let $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_n + 1)$ for $n \geq 1$

a) $S_2 = \frac{2}{3}$ $S_3 = \frac{5}{9}$ $S_4 = \frac{14}{27}$

b) let k be some n that is arbitrarily large

$$S_k = \frac{1}{3}\left(\frac{1}{3}\left(\cdots \frac{1}{3}(1+1) \cdots + 1\right) + 1\right)$$

$$= \frac{1}{3^{k-1}}(1) + \frac{1}{3} + \frac{1}{3^2} + \cdots \frac{1}{3^{k-1}}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{k-2}} + \frac{2}{3^{k-1}}$$

$$\frac{1}{2} - S_k = \frac{1}{2} - \frac{1}{3} - \frac{1}{3^2} - \cdots \frac{2}{3^{k-1}}$$

$$= \frac{3-2}{6} - \frac{1}{3^2} - \cdots \frac{2}{3^{k-1}}$$

$$= \frac{9-6}{2 \cdot 3^3} - \cdots \frac{2}{3^{k-1}}$$

$$= \frac{3}{2 \cdot 3^3} - \cdots \frac{2}{3^{k-1}}$$

$$= \frac{1}{2 \cdot 3^2} - \frac{1}{3^3} \cdots - \frac{2}{3^{k-1}}$$

$$= \frac{3^3 - 2 \cdot 3^2}{2 \cdot 3^5} \cdots - \frac{2}{3^{k-1}}$$

$$= \frac{1}{2 \cdot 3^3} \cdots - \frac{2}{3^{k-1}}$$

$$= \frac{1}{2 \cdot 3^{k-2}} - \frac{2}{3^{k-1}}$$

$$= \frac{3^{k-1} - 2^2 3^{k-2}}{2 \cdot 3^{2k-3}}$$

$$= \frac{3^{k-1} - 2 \cdot 3^{k-2} - 2 \cdot 3^{k-2}}{2 \cdot 3^{2k-3}}$$

$$= \frac{3^{k-2} - 2 \cdot 3^{k-2}}{2 \cdot 3^{2k-3}} < 0$$

$\therefore \quad S_k > \frac{1}{2}$

Show $S_2 > \frac{1}{2}$

$S_2 = \frac{1}{3}(1+1) = \frac{2}{3} > \frac{1}{2}$

$\therefore$ since $S_2 > \frac{1}{2}$ and $S_k > \frac{1}{2}$, S_n and $S_{n+1} > \frac{1}{2}$

$\therefore \forall n \; S_n > \frac{1}{2}$.

c) Show (S_n) is decreasing.

from b) $\frac{1}{2} - S_k = \frac{3^{k-2} - 2 \cdot 3^{k-2}}{2 \cdot 3^{2k-3}}$

assume $n > 1$

$$\frac{1}{2} - S_n = \frac{3^{n-2} - 2 \cdot 3^{n-2}}{2 \cdot 3^{2n-3}}$$

$$S_n - \frac{1}{2} = \frac{2 \cdot 3^{n-2} - 3^{n-2}}{2 \cdot 3^{2n-3}}$$

$$S_n = \frac{3^{n-2}}{2 \cdot 3^{2n-3}} + \frac{1}{2}$$

$$S_n = \frac{1}{2 \cdot 3^{n-1}} + \frac{1}{2}$$

Since the only n is a negative exponent, as n increases, S_n will decrease, thereby making $S_{n+1} < S_n$ and $S_{n+2} < S_{n+1}$ and etc.

d) Show $\lim S_n$ exists and find $\lim S_n$

from c) $S_n = \frac{1}{2 \cdot 3^{n-1}} + \frac{1}{2}$

$$\lim S_n = \frac{1}{2 \cdot 3^{\infty}} + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

10.11 Let $t_1 = 1$ and $t_{n+1} = \left(1 + \frac{1}{4n^2}\right) t_n$ for $n \geq 1$

a) Show that $\lim t_n$ exists.

$$t_{n+1} = \left(1 + \frac{1}{4n^2}\right) t_n$$

$$= \left(\frac{4n^2+1}{4n^2}\right) t_n$$

$$= \left(\frac{4n^2+1}{4n^2}\right)\left(\frac{4(n-1)^2+1}{4(n-1)^2}\right) t_{n-1} \qquad \text{for } n \geq 2$$

$$\vdots$$

$$= \left(\frac{4n^2+1}{4n^2}\right)\left(\frac{4(n-1)^2+1}{4(n-1)^2}\right) \cdots (1)$$

$(n-1)^2$

$= n^2 - 2n + 1$

$n^2 - 4n + 4$

$n^2 - 6n + 9$

$$= \frac{(4n^2+1)(4(n-1)^2+1)(4(n-2)^2+1)\cdots}{4n!^2}$$

$$= \frac{4^n n!^2 + n-1(4n^2 + 4(n-1)^2 + 4(n-2)^2 \ldots) + 1}{4n!^2}$$

$$= \frac{4^{n-1} n!^2 + (n-1)(n^2 + (n-1)^2 + (n-2)^2 \ldots)}{n!^2} + \frac{1}{4n!^2}$$

$$= 4^{n-1} + \underbrace{\frac{(n-1)(n^2 + (n-1)^2 + (n-2)^2 \ldots)}{n!^2}} + \frac{1}{4n!^2}$$

$$\to \frac{(n-1)(n(n^2) - (2n + 4n + 6n \ldots 2(n-1)n) + (1 + 4 + 9 + \ldots (n-1)^2))}{n!^2}$$

$$\frac{(n-1)(n^3) - 2n(n-1)(1 + 2 + 3 + \ldots (n-1)) + (1 + 4 + 9 + \ldots (n-1)^2))}{n!}$$

Using sum of natural numbers & sum of perfect squares equivalencies

$$\frac{(n-1)(n^3) - 2n(n-1)\frac{(n-1)(n)}{2} + \frac{(n-1)(n)(2n-1)}{6}}{n!}$$

$$= \frac{(n-1)(n)\left(n^2 - n(n-1) + \frac{1}{6}(2n-1)\right)}{n!}$$

$$= \frac{n^2 - n(n-1) + \frac{1}{6}(2n-1)}{(n-2)!\,n!}$$

$\frac{1}{4n!^2}$ and $\frac{n^2-n(n-1)+\frac{1}{6}(2n-1)}{(n-2)!\,n!}$ both go to zero as $n \to \infty$

4^{n-1} goes to infinity as $n \to \infty$

$\therefore \quad 0 + \infty + 0 = \infty$

$\therefore \quad \lim t_n = +\infty$

2) Squeeze test. Let a_n, b_n, c_n be three sequences such that $a_n \le b_n \le c_n$ and $L = \lim a_n = \lim c_n$ show that $\lim b_n = L$

if $a_n \le b_n \le c_n$, then $\forall n$, $a_n \le b_n \le c_n$

let $n \to \infty$

$\lim b_n \ge \lim a_n$

therefore $\lim b_n \ge L$

$b_n \le c_n$

$\lim b_n \le \lim c_n$

therefore $\lim b_n \le L$

therefore $L \le \lim b_n \le L$

$L = L$, therefore the only value that satisfies $L \le \lim b_n \le L$ is L

therefore $\lim b_n = L$, $L \le L \le L$