0. 9.9. Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$ (As n gets big, t_n gets bigger than s_n)

a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$

Given that $\lim s_n = +\infty$ we have $\lim s_n = +\infty$ if for each $M > 0$, there is a number N such that

$n > N \rightarrow s_n > M$

~~lim~~ $\lim(\frac{1}{s_n}) = 0$ by 9.10

WTS: $\lim(\frac{1}{s_n}) = 0 \rightarrow \lim(\frac{1}{t_n}) = 0$

WTS: ~~$\exists M > 0$~~

~~Let $\varepsilon > 0$ and let~~

WTS: for each $M > 0$ there exists a N such that $n > N$ implies $t_n > M$.

Let N_0 exist such that $s_n \leq t_n \; \forall n > N_0$

Further suppose $\lim s_n = +\infty$, then for each $M > 0$ there exists a N such that $n > N$ implies $s_n > M$.

Then for every $M > 0$ there exists some $N' \geq N_0$ such that $t_n > M$, because $t_n \geq s_n > M \; \forall n > N_0$.

hence $\lim t_n = +\infty$ by the definition of a limit being infinity. $\square$

$\exists N_0$ such that $s_n \leq t_n$ $\forall n > N_0$

9.9b) Prove if $\lim t_n = -\infty$ then $\lim s_n = -\infty$

$\lim t_n = -\infty$, thus for each $M < 0$, there is an N such that $n > N \rightarrow t_n < M$

WTS: For each M, $\exists N'$ such that for $n > N$, $s_n < M$

For each $M < 0$, there is some N such that $n > N$ and $t_n < M$
Let N_0 exist such that for $\forall n > N_0$ $s_n \leq t_n$, then let N' exist such that $N' \geq N_0$,
$n > N' \rightarrow s_n \leq t_n$
$s_n \leq t_n < M$ $\forall n > N' \geq N_0$
thus $s_n < M$ $\forall n > N'$
and by the definition for a limit being negative infinity,
$\lim s_n = -\infty$ $\square$

a.4c if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$ WTS $\lim s_n - \lim t_n \leq 0$

Let $\lim s_n = s$ and $\lim t_n = t$

Then $\forall \varepsilon > 0$ $\exists N_s$ such that $n > N_s$ implies $|s_n - s| < \varepsilon$

and, $\exists N_t$ such that $n > N_t$ implies $|t_n - t| < \varepsilon$

We have some N_0 such that $s_n \leq t_n$ $\forall n > N_0$

Let ~~$s_n - t_n$~~ be the ~~limit~~ sequence defined by ~~$s_n - t_n$~~ $t_n - s_n$ $\forall n > 0$

WTS: ~~$\exists N_0$~~ $\forall \varepsilon$ $\exists N$ s.t. $\forall n > N \rightarrow$ $|(t_n - s_n) -$ ~~$|s_n - t_0$~~

~~Now: $|(t_n - s_n) - t| < \varepsilon$~~

~~Let $\varepsilon > 0$~~

Let $\varepsilon > 0$ then there exists some $N' \geq N_0$ such that $s_n \leq t_n$ $\forall n \geq N'$

~~which~~ $0 \leq t_n - s_n$ $\forall n \geq N'$

~~$\lim(t_n - s_n)$~~ $\neq$ $|(t_n - s_n) - (t - s)| < \varepsilon$

Q1.5

$\exists N$ such that $\forall n \geq N$

$$\frac{a^{n+1}}{(n+1)!} \leq \frac{a^n}{n!} \qquad \forall a \in A$$

note: $\frac{a^n}{n!} > 0 \quad \forall n$

$\cancel{a^{n+1}}$ $$\frac{a^n \cdot a}{(n+1)n!} \leq \frac{a^n}{n!}$$

$$\frac{a}{n+1} \leq 1$$ For sufficiently large n, as n gets big

thus.

$$\frac{a^{n+1}}{(n+1)!} \leq \frac{a^n}{n!}$$

i.e the sequence is monotonically decreasing and bounded below by zero, thus it converges

thus $\lim \frac{a^n}{n!} = 0$

$\square$

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$

S is bounded $\Rightarrow \sup S \in \mathbb{R}$

$\sup S - \frac{1}{n}$ is not an upperbound of S, and is thus in S, so there is an elem in S denoted s_n, greater than $\sup S - \frac{1}{n}$.

We construct our sequence with this term, with $\sup S - \frac{1}{n}$ as the first term.

We have our sequence s_n with $\sup S > s_n \geq s_1 = \sup S - 1$, thus a sequence ~~exists with~~ s_n ~~$\sup S$~~ s_n exists with $\lim s_n = \sup S$

10.8 Let s_n be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$. Prove σ_n is an increasing sequence.

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \dots + s_n + s_{n+1})$$

$$\sigma_{n+1} - \sigma_n \overset{?}{\geq} 0$$

$$\frac{1}{n+1}(s_1 + s_2 + \dots + s_n + s_{n+1}) \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\frac{1}{n+1}(s_1 + s_2 + \dots + s_n) + \frac{s_{n+1}}{n+1} \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\frac{1}{n+1} + \frac{s_{n+1}}{(n+1)(s_1 + s_2 + \dots + s_n)} \geq \frac{1}{n}$$

$$\frac{1}{n+1}\left(1 + \frac{s_{n+1}}{s_1 + s_2 + \dots + s_n}\right) \geq \frac{1}{n}$$

$$1 + \frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{n+1}{n}$$

$$\frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{n+1}{n} - \frac{n}{n}$$

$$\frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{1}{n} \rightarrow n s_{n+1} \geq s_1 + s_2 + \dots + s_n$$

because s_n is an increasing seq, s_n is greater than all s_i $\forall i \leq n$

thus n successive ~~terms~~ s_{n+1} is bigger than $\sum_{i=1}^{n-1} s_i$

thus σ_n is increasing

10.9 $s_1 = 1$ $s_2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$

$s_3 = \frac{2}{3} \cdot (\frac{1}{2})^2 = \frac{1}{6}$ $s_4 = \frac{3}{4} (\frac{1}{6})^2 = \frac{1}{48}$

b) We show s_n is monotonically decreasing
Proceed via induction
Our base cases are clearly true
$s_1 > s_2 > s_3 > s_4$
Assume this holds for $s_1 \geq s_2 \geq \dots \geq s_n$
Then

$$s_{n+1} = \frac{n}{n+1} s_n^2 = \frac{n}{n+1} \cdot s_n \cdot s_n$$

$s_n^2 \leq 1$ by our base cases

$\frac{n}{n+1} < 1$ clearly

therefore $s_{n+1} < s_n$ by induction.
We now show the sequence is bounded. Because $s_1 = 1$ and the sequence is decreasing, it is bounded above by 1, further, because the sequence consists of products of terms less than 1 and greater than 0, the sequence is bounded below by zero.

10.9 (c). By Thm 10.2 the sequence must converge, we evaluate the limit

$$\lim_{n\to\infty} s_{n+1} = \lim \frac{n}{n+1} \cdot \lim s_n = 0$$