10.6

a) Let (S_n) be a sequence such that

$$|S_{n+1} - S_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$

we can see that $2^{-n} > 0$ for all $n \in \mathbb{N}$

thus the sequence fits the definition of a Cauchy sequence, as it's terms get closer together as n increases. Thus we know (S_n) converges.

b) The case is not the same as $\frac{1}{n}$ diverges.

11.2 Consider the sequences defined as followed:

$$a_n = (-1)^n, \quad b_n = \frac{1}{n}, \quad c_n = n^2, \quad d_n = \frac{6n+4}{7n-3}$$

a) For each sequence, give an example of a monotone subsequence

a_n: all terms with an even n

b_n: whole thing is monotone

c_n: all monotone

d_n: all monotone

b) Set of subsequential limits

a_n: $\{0, 1\}$ c_n: $\{+\infty\}$

b_n: $\{0\}$ d_n: $\{\frac{6}{7}\}$

c) Give lim sup and liminf

a_n: $\limsup a_n = 1$ $\liminf a_n = -1$

b_n: $\limsup b_n = 0$ $\liminf b_n = 0$

c_n: $\limsup c_n = \infty$ $\liminf c_n = \infty$

d_n: $\limsup d_n = \frac{6}{7}$ $\liminf d_n = \frac{6}{7}$

d) a_n: neither, oscillates
b_n: converges to 0
c_n: diverges to $+\infty$
d_n: converges to $6/7$

e) a_n: bounded above by 1, below by -1

b_n: bounded above by 1, below by 0

c_n: unbounded

d_n: bounded above by $\frac{10}{4}$, below by 0

11.3 $s_n = \cos\left(\frac{n\pi}{3}\right)$ $t_n = \frac{3}{4n+1}$, $u_n = \left(\frac{-1}{2}\right)^n$

$v_n = (-1)^n + \frac{1}{n}$

a) $s_n: \{s_{n_k} \; k=6n\}$ $t_n: \{t_{n_k} \mid k=n\}$

$u_n: \{u_{n_k} \mid k=2n\}$ $v_n: \{t_{n_k} \mid k=2n\}$

b) $s_n: \{-1, 1\}$ $t_n: \{0\}$

$u_n: \{0\}$ $v_n: \{-1, 1\}$

c) s_n: $\limsup s_n = 1$, $\liminf s_n = -1$ t_n: $\limsup t_n = 0$, $\liminf t_n = 0$

u_n: $\limsup u_n = 0$, $\liminf u_n = 0$ v_n: $\limsup v_n = 1$, $\liminf v_n = -1$

d. s_n: oscillates t_n: converges to zero

u_n: converge to zero v_n: oscillates

e) s_n: bounded t_n: bounded

u_n: bounded v_n: bounded

11.5 Let (q_n) be an enumeration of all the rationals in the interval $[0,1]$.

a) Give the set of subsequential limits

We will show the set of all subsequential limits is $[0,1]$

First we note s_n is bounded,

Then we show $\forall t \in \mathbb{R}$ there is a subsequence of (q_n) converging to t iff $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$

We will show this set exists for all $t \in [0,1]$

From the denseness of the rationals we know there are infinitely many rationals between any two rationals, thus for any $\varepsilon > 0$ there are infinitely many rationals such that $|s_n - t| < \varepsilon$ is satisfied. Thus, for $\forall t \in [0,1]$, $\{n \in \mathbb{N} : |s_n - t| < \varepsilon\}$ is a infinite set and t is a subsequential limit.

$[0,1]$ is the set of all such limits.

11.5 b)

$$\limsup(a_n) = 1$$

$$\liminf(a_n) = 0$$

What is limsup?

The limsup can be thought of as the largest number that the subsequences or "tails" of a sequence move toward as n reaches infinity, it is not the same as the sup of a sequence.

Consider $S_n = \frac{1}{n}$

We see $\sup(\frac{1}{n}) = 1$ but $\limsup = 0$