Ross 13.3

Let B be the set of all bounded sequences $x = (x_1, x_2, \ldots)$ and define $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \ldots\}$

a) Show d is a metric for B

1)

$$d(x,x) = \sup\{|x_j - x_j| ; j = 1, 2, \ldots\}$$
$$= \sup\{0 ; j = 1, 2, \ldots\}$$
$$= 0 \checkmark$$

2) $d(x,y) = d(y,x)$

$$\sup\{|x_j - y_j| ; j = 1, 2, \ldots\}$$
$$\sup\{|y_j - x_j| ; j = 1, 2, \ldots\}$$

By properties of the absolute value function $|x_j - y_j| = |y_j - x_j|$, so the supremums will be the same $\checkmark$

3) $d(x,z) \leq d(x,y) + d(y,z)$

First we consider $|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$ by the triangle inequality, we take the supr of both sides, showing $\sup|x_j - z_j| \leq \sup|x_j - y_j| + \sup|y_j - z_j|$ for all indices j, thus our definition $d(x,z) \leq d(x,y) + d(y,z)$ is satisfied $\square$

b) how $d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

This distance function allows for infinite distances, which do not provide a valid metric

13.5 Verify one of DeMorgan's Laws for Sets

$$\bigcap\{S\setminus U : U \in \mathcal{U}\} = S\setminus \bigcup\{U : U \in \mathcal{U}\}$$

Proof:

$x \in \bigcap\{S\setminus U : U \in \mathcal{U}\}$

$x \in S\setminus U \ \forall U \in \mathcal{U}$

$x \in S,\ x \notin U \ \forall U \in \mathcal{U}$

$x \in S,\ x \notin \bigcup\{U : U \in \mathcal{U}\}$

$x \in S\setminus \bigcup\{U : U \in \mathcal{U}\}$

as desired.

b. Show the intersection of any collection of closed sets is a closed set.

Given a collection of ~~open~~ closed sets there exists some set $O = X \setminus C$ in some metric space X.
By applying a) we see that any collection of closed sets is closed.

13.7 Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence open intervals.

Let O be an open set in $\mathbb{R}$ s.t. $O \subseteq \mathbb{R}$, let $x \in O$ be an element in O. Then an interval I exists such that $x \in I$ and I is a proper subset of O, if I exists, then there exists a largest interval containing x, the union of all such intervals. These intervals are disjoint and must contain a unique rational, thus every open set is made up of finite intervals I, or a countably infinite number of such intervals.

Recall given (X, d) a metric space, and S a subset of X, we defined the closure of S to be $S^- = \{p \in X \mid \exists (p_n) \in S \text{ s.t. } \lim(p_n) = p\}$

Prove taking the closure again won't make it any bigger.

From the definition of S^- there exists a sequence p_n for all x such that $\lim p_n = p$. The closure of S^- will necessarily be closed by definition, hence there will be some sequence $x_i \in S^{--}$ $\forall i$ ~~that converges to~~ such that x_i is $|x_i - x| < \frac{\varepsilon}{i}$ and converges to p, for all such sequences. Thus $S^{--} = S^-$

5. If $x \in S^-$, the closure of S of some subset of X. Let C be a closed subset of X and S be a subset of C. Let x not be in C, then x is not in S. However, $S^- \subseteq S$, a contradiction, x must be in C for any $S \subseteq C$, thus S^- must be the intersection of all such C closed sets