Hw 6

1). Prove $[0,1]^2$ in $\mathbb{R}^2$ is sequentially compact

Claim: Any two compact sets have a Cartesian product that is compact

Proof: Let $(x_n, y_n) \in X \times Y$ exist such that X and Y are compact.
~~Then ∃ x_{n_k} s.t. for some x∈~~
Then there is a subsequence x_{n_k} that converges to some point $x \in X$ and some subsequence y_{n_k} exists such $y_{n_{k_\ell}}$ that a sub-subsequence converges to some $y \in Y$. Any sub-subsequence of x_{n_k} also converges to x.
Thus,

$$(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \to (x, y)$$

as ℓ approaches ∞, thus

any $X \times Y$ with X and Y compact is compact

2. Let E be the set of points $x \in [0,1]$, whose decimal expansion consist only of 4 and 7. Is E countable? is E compact?

The set E is uncountable. This is shown by drawing a bijection to the set of all ~~infinite ler~~ possible binary strings (0's and 1's) which is shown to be ~~false~~ uncountable by Cantor's Diagonalization argument.

E is Compact
We will show that every infinite subset of E has a limit point in E which implies E is compact.

From weirstrass every bounded infinite subset of $\mathbb{R}^1$ has a limit point in $\mathbb{R}^1$, $[0,1]$ is bounded, thus limit points of subsets of $[0,1]$ and E have limit points in $[0,1]$ thus E is compact

3. Let $A_1, A_2, \ldots$ be subsets of metric space. If $B = \bigcup_i A_i$, then $\overline{B} \supset \bigcup_i \overline{A_i}$. Is it possible that this inclusion is a strict inclusion?

Consider the set of A_i to be covers

$$A_i = \left(\tfrac{1}{i}, 1\right)$$

The infinite union of these subsets

$$B = (0,1)$$

$$\overline{B} = [0,1]$$

We can see that the closure contains zero while no subset of B, A_i contains a zero, thus a strict inclusion exists in this case.

4. Examine the set E from question 2, E is a subset of the closed set [0,1], but is made up of infinitely uncountably number of disjoint subsets.