$$x = \sqrt{4+2\sqrt{3}} - \sqrt{3}$$

$$x+\sqrt{3} = \sqrt{4+2\sqrt{3}}$$

$$(x+\sqrt{3})^2 = 4+2\sqrt{3}$$

$$x^2+2\sqrt{3}x+3 = 4+2\sqrt{3}$$

$$x^2+2\sqrt{3}x-1-2\sqrt{3} = 0$$

$$x^2+2\sqrt{3}x-(1+2\sqrt{3}) = 0$$

rational solutions: $\frac{p}{q} = \frac{\pm(1+2\sqrt{3})}{\pm 1}$

$$= \pm(1+2\sqrt{3})$$

$$(x+\sqrt{3})(x+\sqrt{3})$$

$$x^2+\sqrt{3}x+\sqrt{3}x+3$$

$$x^2+2\sqrt{3}x+3$$

$$2\sqrt{3}\cdot(-\sqrt{3})$$

$$4+2\sqrt{3} - 2\sqrt{3}\sqrt{4+2\sqrt{3}} + 3 + 2\sqrt{3}\left(\sqrt{4+2\sqrt{3}} - \sqrt{3}\right) + 3 = 4+2\sqrt{3}$$

$$4+2\sqrt{3} - \cancel{2\sqrt{3}\sqrt{4+2\sqrt{3}}} + 3 + \cancel{2\sqrt{3}\sqrt{4+2\sqrt{3}}} - 6 + 3 = 4+2\sqrt{3}$$

$$4+2\sqrt{3} + \cancel{6-6} = 4+2\sqrt{3}$$

$$4+2\sqrt{3} = 4+2\sqrt{3}$$ ✓

x is a rational solution

x is rational

2.7b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

$x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$

$x + \sqrt{2} = \sqrt{6+4\sqrt{2}}$

$(x+\sqrt{2})^2 = 6+4\sqrt{2}$

$x^2 + 2\sqrt{2}x + 2 = 6+4\sqrt{2}$ plug $x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$

$6+4\sqrt{2} - 2\sqrt{2}\sqrt{6+4\sqrt{2}} + 2 + 2\sqrt{2}\left(\sqrt{6+4\sqrt{2}} - \sqrt{2}\right) + 2 \stackrel{?}{=} 6+4\sqrt{2}$

$6+4\sqrt{2} - \cancel{2\sqrt{2}\sqrt{6+4\sqrt{2}}} + 2 + \cancel{2\sqrt{2}\sqrt{6+4\sqrt{2}}} - 4 + 2 \stackrel{?}{=} 6\,4\sqrt{2}$

$6+4\sqrt{2} \cancel{+4} \cancel{-4} \stackrel{?}{=} 6+\sqrt{2}$

$6+4\sqrt{2} = 6+4\sqrt{2}$ ✓

is a rational solution

thus $\sqrt{6+4\sqrt{2}} - \sqrt{2}$ is rational

3.6 a) Prove $|a+b+c| \le |a|+|b|+|c| \quad \forall a,b,c \in \mathbb{R}$

by Δ ineq $\quad |a+b| \le |a|+|b| \quad |b+c| \le |b|+|c|$

$$|a+b+c|$$

$$= |(a+b)+c| \le |a+b|+|c| \le |a|+|b|+|c|$$

thus $\quad |a+b+c| \le |a|+|b|+|c|$

$\square$

4.11 Consider $a, b \in \mathbb{R}$ where $a < b$. Use Densness of $\mathbb{Q}$ 4.7 to show there are infinitely many rationals between a and b

Let a, b be two real numbers with $a < b$

then by the Densness of $\mathbb{Q}$ some rational r exists such that $a < r < b$ where $r = \frac{m}{n}$ $m, n \in \mathbb{Z}$

Now, consider the fact we have two pairs of real numbers where $a < r$ and $r < b$, using the Completeness of $\mathbb{Q}$ yet again shows that two more rationals r' and r'' exist such that $a < r' < r < r'' < b$, following this pattern, infinitely many rationals exist between a and b. $\square$

4.14. Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let $A+B$ be the set of all sums $a+b$ where $a \in A, b \in B$

$\geq$) $\sup(A) \geq a \ \forall a \in A$

$\sup B \geq b \ \forall b \in B$

$\sup(A) + \sup B \geq a+b \ \forall a \in A \ \forall b \in B$

$a+b \ \forall a \in A, \forall b \in B$ is defined to be the set $A+B$ as desired we also know $\sup(A+B)$ exists because $(A+B)$ is a bounded subset of $\mathbb{R}$, thus

$\sup A + \sup B \geq \sup(A+B)$

Both sides of the inequality have been shown, thus

$\sup A + \sup B = \sup(A+B)$

a) $\leq$) Claim: $\forall b \in B$, $\sup(A+B) - b$ is an upperbound of A

WTS: $\sup(A+B) - b \geq a$

$\sup(A+B) \geq a+b \quad \forall a \in A, \forall b \in B$

$A+B$ is defined as the set of all sums $a+b$ and neccessarily the inequality is true.

Thus $\sup(A+B) - b \geq a \quad \forall a \in A$

$\rightarrow \sup(A+B) - b \geq \sup A$

$\rightarrow \sup(A+B) - \sup A \geq b \quad \forall b \in B$

$\rightarrow \sup(A+B) - \sup A \geq \sup B$

$\rightarrow -\sup(A) - \sup B \geq -\sup(A+B)$

$\rightarrow \sup(A) + \sup B \leq \sup(A+B)$

4.14 b) Prove $\inf(A+B) = \inf A + \inf B$

$\leq$) $\inf(A+B) \leq a+b \quad \forall a \in A, \forall b \in B$ by def

$\inf(A+B) - b \leq a \quad \xrightarrow{\forall a \in A} \quad \inf(A+B) - b \leq \inf A$

Similarly, $\inf(A+B) - a \leq b \rightarrow \inf(A+B) - a \leq \inf B$

$\rightarrow \inf(A+B) \leq \inf A + \inf B$

$\geq$) $\inf A \leq a \quad \forall a \in A$, $\inf B \leq b \quad \forall b \in B$

$A+B$ is the set made of all sums of $a+b$ from A and B.

So the greatest lower bound for $A+B$ must be greater or equal the bounds of our two subsets

$\inf(A+B) \geq \inf A + \inf B$

thus

$\inf(A+B) = \inf A + \inf B$

7.5 Determine Limits

a) $\lim S_n$ $\quad S_n = \sqrt{n^2+1} - n \quad \dfrac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n}$

$$= \frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n}$$

$$= \lim \frac{1}{\sqrt{n^2+1}+n}$$

$$= \frac{\lim 1}{\lim \sqrt{n^2+1}\ n} = \frac{1}{\infty} = 0$$

c) $\lim(\sqrt{4n^2+n} - 2n) \left(\dfrac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}\right)$

$$\lim \frac{\cancel{4n^2}+n-\cancel{4n^2}}{\sqrt{4n^2+n}+2n} = \lim \frac{n}{\sqrt{4n^2+n}+2n}$$

$$= \lim \frac{1}{\frac{\sqrt{4n^2+n}}{n}+2}$$

$$\lim \frac{\sqrt{4n^2+n}}{n}+2 = \lim \sqrt{\frac{4n^2+n}{n^2}}+2$$

$$= \lim \sqrt{4+\tfrac{1}{n}}+2 \qquad \lim \tfrac{1}{n} = 0$$

$$= \lim \sqrt{4+0}+2$$

$$= 2+2 = 4$$

$$= \lim \frac{1}{4} = \frac{1}{4}$$

b) $\lim(\sqrt{n^2+n} - n)$

$$= \lim \left((\sqrt{n^2+n}-n)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)\right)$$

$$= \lim \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n}$$

$$= \lim \frac{1}{\frac{1}{n}\sqrt{n^2+n}+1}$$

$$= \frac{\lim 1}{\lim \frac{\sqrt{n^2+n}}{n}+1}$$

$$\lim \frac{\sqrt{n^2+n}}{n}+1$$

$$= \lim \sqrt{\frac{n^2+n}{n^2}}+1$$

$$= \lim \sqrt{1+\tfrac{1}{n}}+1$$

$$\lim \tfrac{1}{n} = 0$$

$$= \lim(\sqrt{1}+1) = 2$$

$$= \frac{\lim 1}{\lim 2} = \frac{1}{2}$$

1.10

$$\sum_{i=1}^{n} 2n + (2i-1) = 3n^2$$

Proof via induction

Base Case: $n=1$

$$2(1) + 2(1) - 1 = 3(1)^2$$

$$2 + 2 - 1 = 3 = 3 \checkmark$$

Inductive step:

$$\sum_{i=1}^{n+1} 2(n+1) + 2i - 1$$

$$= \sum_{i=1}^{n} \left(2(n+1) + 2i - 1\right) + 4n + 2 + 2(n+1) - 1$$

$$= \sum_{i=1}^{n} \left(2(n+1) + 2i - 1\right) + 4n + 2 + 2n + 2 - 1$$

$$= \sum_{i=1}^{n} \left(2(n+1) + 2i - 1\right) + 6n + 3 \stackrel{?}{=} 3(n+1)^2$$

By I.H.

$$= 3n^2 + 6n + 3 = 3[n^2 + 2n + 1]$$

$$= 3n^2 + 6n + 1$$

$\square$

1.13

For $n \in \mathbb{N}$, let $n!$ denote product of n consecutive terms $1 \cdot 2 \cdot 3 \cdots n$

Let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } k = 0, 1, \ldots, n.$$

Binomial thm asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$= a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

a) Verify bin thm for $n = 1, 2, 3$

$n = 1$

$$(a+b)^1 = a+b = \binom{1}{0}a^1 + \binom{1}{1}a^0 b$$

$$= a+b \checkmark$$

$n = 2$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\stackrel{?}{=} \binom{2}{0}a^2 + \binom{2}{1}a^1 b + \binom{2}{2}a^0 b^2$$

$$= a^2 + 2ab + b^2 \checkmark$$

Similarly ... $k!(n+1-k)(n-k)$

From
1.12

a) $n=3$

$(a^2+2ab+b^2)(a+b) = a^3+2a^2b+ab^2$
$+a^2b+2ab^2+b^3$
$= a^3+3a^2b+3ab^2+b^3$

$\stackrel{?}{=} \binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3$
$= a^3+3a^2b+3ab^2+b^3$ ✓

b) Show $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ for $k=1,2,\dots,n$

Base case: $k=1$

$\binom{n}{1}+\binom{n}{0}=\binom{n+1}{1}$

$\frac{n!}{1!(n-1)!}+\frac{n!}{0!\,n!}=\frac{(n+1)!}{1!\,n!}$

$\frac{n\cdot\cancel{(n-1)!}}{\cancel{(n-1)!}}+1=\frac{(n+1)\cancel{n!}}{\cancel{n!}}$

$n+1=n+1$ ✓

Assume $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ holds for all k, in the $k+1$ case we have

$\binom{n}{k+1}+\binom{n}{k}\stackrel{?}{=}\binom{n+1}{k+1}$

Algebraically: $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$

$\frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k-1)!}\stackrel{?}{=}\frac{(n+1)!}{k!(n+1-k)!}$

$\frac{n!}{k!(n-k)\cdot(n-k-1)!}+\frac{n!}{(k-1)!(n-k-1)!}=\frac{(n+1)!}{k!(n+1-k)(n-k)(n-k-1)}$

$\frac{n!}{k!(n-k)}+\frac{n!}{(k-1)!}=\frac{(n+1)!}{k!(n+1-k)(n-k)}$
:

Bam

$$\frac{n!}{k!(n-k)} + \frac{n!}{(k-1)!} = \frac{n\cdot(n)!}{k!(n+1-k)(n-k)}$$

$$\frac{1}{k!(n-k)} + \frac{1}{(k-1)!} = \frac{n}{k!(n+1-k)(n-k)}$$

$$\frac{\cancel{k!}}{\cancel{k!}(n-k)} + \frac{k\cdot\cancel{(k-1)!}}{\cancel{(k-1)!}} = \frac{n}{(n+1-k)(n-k)}$$

$$\frac{1}{n-k} + k = \frac{n}{(n+1-k)(n-k)}$$

$$1 + k(n-k) = \frac{n}{(n+1-k)}$$

$$1 + kn - k^2 = \frac{n}{n+1-k}$$

$$(1+kn-k^2)(n+1-k) = n$$

$$n+1-k+kn^2+kn-k^2n-k^2n-k^2+k^3 = n$$

$$n+1-k+kn^2+kn-k^2n-k^2n-k^2+k^3 = n$$

$$n+1-k+kn^2+kn-2k^2n-k^2+k^3 = n$$

c) Prove bin thm using induction and b)

Prove:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$= a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \dots + nab^{n-1} + b^n$$

Base case: $n=1$

$$(a+b)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^0 b$$

$$a+b = a+b \checkmark$$

Assume $P(n)$ holds for all n

For $P(n+1)$ we have

$$(a+b)^{n+1} = \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^{n-2}b^2 + \dots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$$

$$(a+b)(a+b)^n \overset{(a+b)}{=} \left(a^{n} + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n-1}ab^{n-1} + b^n\right)$$

$$= a^{n+1} + \left(1 + \binom{n}{1}\right)a^n b + \left(\binom{n}{1} + \binom{n}{2}\right)a^{n-1}b^2 + \dots \left(\binom{n}{n-1} + 1\right)ab^n + b^{n+1}$$

From b) we know $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$

thus every coefficient is as desired.

thus $n+1$ case is true.

$\square$

$n-(k-1)$

$n-k+1$

Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$$\frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} = \frac{(n+1)!}{k!\,(n+1-k)!}$$

$$\frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} = \frac{(n+1)\cdot n!}{k!\,(n+1-k)!}$$

$$\frac{1}{k!\,(n-k)!} + \frac{1}{(k-1)!\,(n+1-k)!} = \frac{n+1}{k!\,(n+1-k)!} = \frac{n+1}{k!\,(n+1-k)(n-k)!}$$

$$\frac{(n+1-k)\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}} + \frac{\cancel{(n+1-k)(n-k)!}}{(k-1)!\,\cancel{(n+1-k)!}} = \frac{n+1}{k!}$$

$$\frac{n+1-k}{k!} + \frac{1}{(k-1)!} \quad \frac{n+1}{k!}$$

$$\cancel{n+1-k + \frac{k!}{k-1} = n+1}$$

$$\cancel{n+1-k + k\cdot(k-2)! = n+1}$$

$$\cancel{\frac{k!}{k-1}} \quad \cancel{k\cdot(k-1)\cdot(k-2)!}$$

$$n+1-k + \frac{k!}{(k-1)!} = n+1$$

$$n+1-\cancel{k}+\cancel{k} = n+1$$

$$n+1 = n+1 \checkmark$$

2.1 Show that $\sqrt{3}, \sqrt{5}, \sqrt{7}$, and $\sqrt{31}$ are not rationals

$\sqrt{3}$ is not rational

Pf: By rational zeroes Thm,

$x^2 - 3 = 0$, here $n=2$, $c_2 = 1$, $c_1 = 0$, and $c_0 = -3$

The only rational solutions could be ± 1 and ± 3, none of which are solutions to the polynomial, thus $\sqrt{3}$ is not rational

$\sqrt{5}$

Similarly, only rational solutions of $x^2 - 5$ are ± 1 and ± 5,

$\sqrt{7}$

Similarly, only possible rational solutions of $x^2 - 7$ are ± 1 and ± 7

$\sqrt{24}$

Similarly only possible rational solutions ~~of~~ $x^2 - 24$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$, none of which are solutions hence $\sqrt{24}$ is irrational

Similarly for $\sqrt{31}$

2.2 show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are irrational

$\sqrt[3]{2}$

$x^3 - 2$

$\frac{p}{q} = \frac{\pm 1, \pm 2}{\pm 1}$, only possible rational solutions are ± 1 , ± 2 , none of which satisfy the polynomial, $\sqrt[3]{2}$ is irrational

$\sqrt[7]{5} \rightarrow x^7 - 5$, only possible rational solutions are ± 1, ± 5 , none of which are solutions, $\sqrt[7]{5}$ is irrational

$\sqrt[4]{13} \rightarrow x^4 - 13$, only possible rational solutions are ± 1, ± 13 , none of which are solutions, $\sqrt[4]{13}$ is irrational