Math 104 HW 10

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8/27/2020

Ross 33.4

Using the hint:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in R \setminus \mathbb{Q} \end{cases}$$
$$|f(x)| = 1$$

Ross 33.7

(a) Then for any partition P made of intervals $\{I_i = [x_i, x_{i+1}]\}_{i=1}^n$, define

$$\Delta x_i = x_{i+1} - x_i$$
$$M_i(f) = \sup_{x \in I_i} f(x)$$
$$m_i(f) = \sup_{x \in I_i} f(x)$$
$$U(f^2, P) - L(f^2, P) = \sum_{i=1}^n (M_i(f^2) - m_i(f^2))\Delta x_i$$

By compactness of each interval, $M_i(f^2) = f(x)^2$ for some $x \in I_i$ and $m_i(f^2) = f(y)^2$ for some $y \in I_i$. Therefore

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{i=1}^n (f(x)^2 - f^2(y))\Delta x_i \\ &= \sum_{i=1}^n (f(x) + f(y))(f(x) - f(y))\Delta x_i \\ &\leq 2B \left(\sum_{i=1}^n (f(x) - f(y))\Delta x_i \right) \\ &\leq 2B \left(\sum_{i=1}^n \sup_{x,y \in I_i} (f(x) - f(y))\, \Delta x_i \right) \\ &\leq 2B\, (U(f, P) - L(f, P)) \end{aligned}$$

(b) Well given any $\varepsilon > 0$, we can get a partition P such that

$$U(f, P) - L(f, P) \leq \frac{\varepsilon}{2B}$$
$$U(f^2, P) - L(f^2, P) \leq 2B\, (U(f, P) - L(f, P)) = \varepsilon$$

Ross 33.13

We can use the intermediate value theorem for integrals on $f - g$. For some $x \in (a, b)$:

$$f(x) - g(x) = \frac{1}{b-a} \int_a^b f - g = \frac{1}{b-a} \left(\int_a^b f - \int_a^b g \right) = 0$$
$$f(x) = g(x)$$

Ross 35.4

By theorem 35.13, we can integrate with respect to $F'(x)dx = \cos(x)dx$

(a)

$$\int_0^{\pi/2} x\cos(x)dx = x\sin(x) - \int \cos(x)dx \Big|_0^{\pi/2}$$
$$= x\sin(x) - \sin(x)|_0^{\pi/2}$$
$$= \pi/2 - 1$$

(b)

$$\int_0^{\pi/2} x\cos(x)dx = x\sin(x) - \sin(x)|_{-\pi/2}^{\pi/2}$$
$$= (\pi/2 - 1) - (\pi/2 - 1)$$
$$= 0$$

Ross 35.9a

This follows by the same argument as theorem 33.9, the intermediate value theorem for integrals. Let $M = \max_{x \in [a,b]} f(x)$ and $m = \min_{x \in [a,b]} f(x)$. If $M = m$ then f is constant in which case any x works. If $m < M$, then $m \leq f \leq M$ and then by theorem 35.9:

$$\int_a^b mdF \leq \int_a^b fdF \leq \int_a^b M$$
$$m \leq \frac{1}{F(b) - F(a)} \int_a^b ddF \leq M$$

And then by the intermediate value theorem for continuous functions, $f(x) = \frac{1}{F(b) - F(a)} \int_a^b ddF$ for some $x \in (a, b)$ as we have shown it is an intermediate value.