Math 104 HW 11

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Ross 34.2

(a) By the fundamental theorem of calculus

$$F(x) = \int_0^x e^{t^2} dt$$
$$\lim_{x\to 0} \frac{\int_0^x e^{t^2} dt}{x} = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0}$$
$$= F'(0^+)$$
$$= e^{0^2} = 1$$

As e^{t^2} is continuous.

(b) By the same argument as before:

$$G(x) = \int_3^{3+x} e^{t^2} dt$$
$$\lim_{h\to 0} \frac{\int_3^{3+h} e^{t^2} dt}{h} = \lim_{h\to 0} \frac{G(h) - G(0)}{h - 0}$$
$$= G'(0)$$
$$= e^9$$

Ross 34.5

Then consider for any x, $[x+1, x-1] \in (a, b)$ for some $a, b \in \mathbb{R}$. Then $F(x)$ can be broken into two integrals and because f is continuous, we can apply the fundamental theorem of calculus to differentiate $F(x)$:

$$F(x) = \int_a^{x+1} f(t)dt - \int_a^{x-1} f(t)dt$$
$$F'(x) = f(x+1) - f(x-1)$$

Ross 34.7

$$
\begin{aligned}
u(x) &= 1 - x^2 \\
u' &= -2x \\
u(0) &= 1 \\
u(1) &= 0 \\
\int_0^1 x\sqrt{1-x^2}dx &= \int_0^1 \frac{-1}{2}u'(x)\sqrt{u}dx \\
&= \int_1^0 \frac{-1}{2}\sqrt{u}u \\
&= \frac{1}{2}\left(\frac{2}{3}1^{3/2}\right) \\
&= \frac{1}{3}
\end{aligned}
$$

Rudin 15

Using integration by parts for $xf(x)$ and $f'(x)$ we get:

$$
\begin{aligned}
\int_a^b xf(x)f'(x)dx &= xf(x)^2\Big|_a^b - \int_a^b (f(x) + xf'(x))\, f(x)dx \\
&= bf(b)^2 - af(a)^2 - \int_a^b f^2(x)dx - \int_a^b xf'(x)f(x)dx \\
&= 0 - 0 - 1 - \int_a^b xf'(x)f(x)dx \\
2\int_a^b xf'(x)f(x)dx &= -1
\end{aligned}
$$

Then using holder's inequality:

$$
\frac{1}{4} = \left|\int_a^b (xf(x))\, f'(x)dx\right|^2 \leq \left|\int_a^b f'(x)^2dx\right|\left|\int_a^b x^2f(x)^2dx\right|
$$

Then if equality were attained for the holder inequality, we would conclude $xf(x)$ and $f'(x)$ are linearly dependent in $L^2[a,b]$ and therefore $cxf(x) = f'(x)$ for $c \in [a,b]$.

$$
\begin{aligned}
f(x) &= ke^{cx^2/2} \\
f'(x) &= cxke^{cx^2/2} = cxf(x)
\end{aligned}
$$

Rudin 16

(a)

$$s\int_1^N \frac{[x]}{x^{s+1}}dx = s\sum_{k=1}^{N}\int_k^{k+1}\frac{k}{x^{s+1}}dx$$
$$= s\sum_{k=1}^{N} -\frac{1}{s}\frac{k}{x^s}\Big|_k^{k+1}$$
$$= \sum_{k=1}^{N} k(k^{-s} - (k+1)^{-s})$$

Then using summation by parts (rudin theorem 3.41), in the form found on wikipedia:

$$\sum_{k=1}^{N} -k((k+1)^{-s} - k^{-s}) = -N(N+1)^{-s} + 11^{-s} + \sum_{k=2}^{N} k^{-s}(1)$$
$$= \left(\sum_{k=1}^{n}\frac{1}{k^s}\right) - \frac{N}{(N+1)^s}$$

And the term $\frac{N}{(N+1)^s} \to 0$ as $N \to \infty$, showing the desired inequality.

(b) First, to show the integral converges for all $s > 0$, simply note that $0 \leq x - [x] \leq 1$ and therefore

$$0 \leq \int_1^\infty \frac{x - x}{x^{s+1}}dx \leq \int_1^\infty \frac{1}{x^{s+1}}dx = \frac{-1}{sx^s}\Big|_1^\infty = \frac{1}{s}$$

Second, to show the identity holds

$$s\int_1^\infty \frac{[x]}{x^{s+1}}dx = s\int_1^\infty \frac{x-(x-[x])}{x^{s+1}}dx$$
$$= s\int_1^\infty \frac{x}{x^{s+1}}dx - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx$$
$$s\int_1^\infty \frac{x}{x^{s+1}}dx = s\int_1^\infty \frac{1}{x^s}dx = \frac{s}{s-1}$$

And by part a), this is sufficient.

Extra one

$$\alpha(1) = \sum_{1<n} 2^{-n} = 1$$
$$\frac{1}{n} < \frac{1}{k} \iff k < n$$
$$\alpha(1/k) = \sum_{k<n} 2^{-n} = 2^{-k}$$

Then consider $x \in \left(\frac{1}{k}, \frac{1}{k-1}\right)$. Then

$$x < \frac{1}{k-1} \implies k - 1 < \frac{1}{x}$$
$$\frac{1}{n} < x \implies \frac{1}{x} < n$$
$$x > \frac{1}{k+1}$$
$$\alpha(x) = \alpha(1/k)$$

And therefore if $x \leq \frac{1}{n}$ we have $\alpha(x) \leq \alpha(1/n) = 2^{-n}$ so for any $\varepsilon > 0$ we can choose $x \leq \log_2(\varepsilon)$ and therefore $\alpha(x) \leq \varepsilon$. So α is continuous at $x = 0$.
Then because $f(x)$ is continuous everywhere except $x = 0$, f is integrable with respect to α. There might be some other conditions, but I can't find the theorem in Ross or Rudin.