HW 1

1.10 Prove $(2n+1)+(2n+3)+(2n+3)+\cdots+(4n-1)=3n^2$ for all positive integers n.

Induction : $P(n)=(2n+1)+(2n+3)+(2n+3)+\cdots+(4n-1)=3n^2$

Basis: $n=1$ $\quad 2(1)+1=3(1)^2=3$ ✓

Induction: Assume true $n=k$ $\quad (2k+1)+(2k+3)+(2k+3)+\cdots+(4k-1)=3k^2$

Show $n=k+1$

$$\underbrace{(2k+1)+(2k+3)+(2k+3)+\cdots+(4k-1)}_{3k^2}+(k+1)=3k^2+(k+1)$$

$$3k^2+k+1 = 3k^2+k+1$$

Then $P(k+1)$ is true

thus, $P(n)$ is true for all $n\in\mathbb{Z}^+$.

1.12 $n! = 1\cdot 2\cdot 3\cdot \cdots n$

$0! = 1$

$\binom{n}{k}=\dfrac{n!}{k!\,(n-k)!}$ for $k=0,1,\cdots,n$

binomial Thm

$$(a+b)^n=\binom{n}{0}a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{n-1}ab^{n-1}+\binom{n}{n}b^n$$
$$=a^n+na^{n-1}b+\tfrac{1}{2}n(n-1)a^{n-2}b^2+\cdots+nab^{n-1}+b^n$$

(a) verify $n=1,2$, and 3

$n=1$: $(a+b)^1=\binom{1}{0}a+\binom{1}{1}a^0b=a+b$

$n=2$: $(a+b)^2=\binom{2}{0}a^2+\binom{2}{1}a^{2-1}b+\binom{2}{2}a^{2-2}b^2=a^2+2ab+b^2$

$n=3$: $(a+b)^3=\binom{3}{0}a^2+\binom{3}{1}a^2b+\binom{3}{2}a^1b^2+\binom{3}{3}a^0b^3=a^2+3a^2b+3ab^2+b^3$

b) show $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ $\quad k=1,2,\cdots,n$ $\qquad \binom{n+1}{k}=\dfrac{(n+1)!}{k!\,((n+1)-k)!}$

$$\binom{n}{k}+\binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!}{k!(n-k)!}\cdot\frac{n-k+1}{n-k+1} + \frac{n!}{(k-1)!(n-k+1)!}\cdot\frac{k}{k}$$

$$= \frac{n!(n-k+1+k)}{k!(n-k+1)!}$$

$$= \frac{(n+1)n!}{k!(n+1-k)!}$$

$$= \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \binom{n+1}{k}$$

c) Prove w/ induction and (b)

Pf by Induction :

$$P(n) = (a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

<u>Basis</u> : $n=1$ $\quad (a+b)^1 = \binom{1}{0}a + \binom{1}{1}a^0 b = a+b$ ✓

<u>Induction</u> : Assume true $n=k$

show true $n=k+1$

$$(a+b)^{k+1} = \binom{k+1}{0}a^n + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

$$= (a+b)\cdot(a+b)^k$$

$$= (a+b)\left[\overset{1}{\cancel{\binom{k}{0}}}a^k + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \cdots + \binom{k}{k-1}ab^{k-1} + \overset{1}{\cancel{\binom{k}{k}}}b^k\right]$$

$$= \Big[\binom{k}{0}a^{k+1} + \binom{k}{1}a^k b + \binom{k}{2}a^{k-1}b^2 + \cdots + \binom{k}{k-1}a^2b^{k-1} + \binom{k}{k}ab^k$$

$$+ \binom{k}{0}a^k b + \binom{k}{1}a^{k-1}b^2 + \binom{k}{2}a^{k-2}b^3 + \cdots + \binom{k}{k-1}ab^k + \binom{k}{k}b^{k+1}\Big]$$

$$= \binom{k}{0}a^{k+1} + \left[\binom{k}{1}+\binom{k}{0}\right]a^k b + \left[\binom{k}{2}+\binom{k}{1}\right]a^{k-1}b^2 + \cdots + \left[\binom{k}{k}+\binom{k}{k-1}\right]ab^k + \binom{k}{k}b^{k+1}$$

formula $\binom{n}{k}+\binom{n}{k-1} = \binom{n+1}{k}$

$$= \binom{k}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k}{k}b^{k+1}$$

formula $\binom{n}{0}=1=\binom{n+1}{0}$ $\quad \binom{n}{n}=1=\binom{n+1}{n+1}$

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2 + \cdots + \binom{k+1}{k}ab^k + \binom{k+1}{k+1}b^{k+1}$$

$\therefore$ P(n+1) is true. Thus, P(n) is true.

2.1 Show $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, $\sqrt{31}$ are not rational #s

Rational Zeros Thm : $x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0 = 0$

solns must be an integer factor of C_0.

$\sqrt{3}$ $x^2 - 3 = 0$

$n=2$, $C_2 = 1$, $C_1 = 0$, $C_0 = -3$

Factors of $C_0 = -3$: $\pm 1, \pm 3$

By Rational zeros thm, only solns

$(\pm 1)^2 - 3 \neq 0$ $(\pm 3)^2 - 3 \neq 0$

$\therefore \sqrt{3}$ is not a rational number $\square$

$\sqrt{5}$ $x^2 - 5 = 0$

$n=2$, $C_2 = 1$, $C_1 = 0$, $C_0 = -5$

Factor of $C_0 = -5$: ± 1, ± 5

By RZT, only solns

$(\pm 1)^2 - 5 \neq 0$ $(\pm 5)^2 - 5 \neq 0$

$\therefore \sqrt{5}$ is not a rational # $\square$

$\sqrt{7}$ $x^2 - 7 = 0$

$n=2$, $C_2 = 1$, $C_1 = 0$, $C_0 = -7$

Factor of $C_0 = -7$: ± 1, ± 7

By RZT, only solns

$(\pm 1)^2 - 7 \neq 0$ $(\pm 7)^2 - 7 \neq 0$

$\therefore \sqrt{7}$ is not a rational # $\square$

$\sqrt{31}$ $x^2 - 31 = 0$

$n=2$, $C_2 = 1$, $C_1 = 0$, $C_0 = -31$

Factor of $C_0 = -31$: ± 1, ± 31

By RZT, only solns

$(\pm 1)^2 - 31 \neq 0$ $(\pm 31)^2 - 31 \neq 0$

$\therefore \sqrt{31}$ is not a rational # $\square$

$\sqrt{24}$ $x^2 - 24 = 0$

$n=2$, $C_2 = 1$, $C_1 = 0$, $C_0 = -24$

Factor of $C_0 = -24$: $\pm 1, \pm 24, \pm 2, \pm 12, \pm 3, \pm 8, \pm 4, \pm 6$

By RZT, only solns

$(\pm 1)^2 - 24 \neq 0$ $(\pm 24)^2 - 24 \neq 0$ $\cdots$ $(\pm 6)^2 - 6 \neq 0$

$\therefore \sqrt{24}$ is not a rational # $\square$

2.2 Show $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational #s

$\sqrt[3]{2}$ $x^3 - 2 = 0$

$n=3$, $C_3 = 1$, $C_2 = 0$, $C_1 = 0$, $C_0 = -2$

Factors of $C_0 = -2$: ± 1, ± 2

By RZT, only possible solns ↗

$(\pm 1)^3 - 2 \neq 0$ $(\pm 2)^3 - 2 \neq 0$

$\therefore \sqrt[3]{2}$ not a rational #. $\square$

$\sqrt[7]{5}$ $x^7 - 5 = 0$

$n=7$, $C_7 = 1$, $C_6 = C_5 = C_4 = C_3 = C_2 = C_1 = 0$, $C_0 = -5$

Factor of $C_0 = -5$: ± 1, ± 5

By RZT, only solns ↗

$(\pm 1)^7 - 5 \neq 0$ $(\pm 5)^7 - 5 \neq 0$

$\therefore \sqrt[7]{5}$ not a rational #. $\square$

$\sqrt[4]{13}$ $x^4 - 13 = 0$

$n=4$, $C_4 = 1$, $C_3 = C_2 = C_1 = 0$, $C_0 = -13$

Factor of $C_0 = -13$: ± 1, ± 13

By RZT, only solns ↗

$(\pm 1)^4 - 13 \neq 0$ $(\pm 13)^4 - 13 \neq 0$

$\therefore \sqrt[4]{13}$ not rational #. $\square$

2.7 Show they're actually rational #s

a) $\sqrt{4+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{1+3+2\sqrt{3}} - \sqrt{3}$ $\quad \sqrt{(\cdots)^2}$

$= \sqrt{1+(\sqrt{3})^2+2\sqrt{3}} - \sqrt{3}$

$= \sqrt{(1+\sqrt{3})^2} - \sqrt{3}$

$= 1+\sqrt{3} - \sqrt{3}$

$= 1 \quad \in \mathbb{Q} \quad \square$

b) $\sqrt{6+4\sqrt{2}} - \sqrt{2}$

$= \sqrt{2+4+4\sqrt{2}} - \sqrt{2}$ $\quad (a+b)^2 = a^2 + 2ab + b^2$

$= \sqrt{(\sqrt{2})^2 + 2^2 + 2\cdot 2\cdot\sqrt{2}} - \sqrt{2}$

$= \sqrt{(\sqrt{2}+2)^2} - \sqrt{2}$

$= \sqrt{2}+2-\sqrt{2}$

$= 2 \quad \in \mathbb{Q} \quad \square$

3.6 a) Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a,b,c \in \mathbb{R}$.

Hint: (triangle inequality twice)

$|a+b| \leq |a| + |b|$ $\quad$ apply $\Delta \neq$

$|a+(b+c)| \leq |a| + |b+c|$ $\quad |b+c| \leq |b|+|c|$

$|a+(b+c)| \leq |a| + |b| + |c|$ $\quad$ apply $\Delta\neq$ twice

$|a+b+c| \leq |a| + |b| + |c|$

b) Use induction to prove $|a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$

$P(n): \ |a_1+a_2+\cdots+a_n| \leq |a_1|+|a_2|+\cdots+|a_n|$

$P(1) = a_1 \leq |a_1| \quad \checkmark$

assume $P(n)$ true.

show $P(n+1)$: $|\underbrace{a_1+a_2+\cdots+a_n}+a_{n+1}| \leq |(a_1+a_2+\cdots+a_n)+a_{n+1}|$

triangle inequality

$\leq |(a_1+a_2+\cdots+a_n)| + |a_{n+1}|$

$\overset{P(n)}{\leq} |a_1|+|a_2|+\cdots+|a_n| + |a_{n+1}|$

$\therefore P(n+1)$ is true.

Thus, $P(n)$ is true.

4.11 The Completeness Axiom

Use Denseness of $\mathbb{Q}$ 4.7 to show ∞ many rationals btw a and b.

If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ st $a < r < b$.

By Deneseness of $\mathbb{Q}$, given $a, b \in \mathbb{R}$ and $a < b$,
then there is a rational $r_1 \in \mathbb{Q}$ st. $a < r_1 < b$.
. . . $r_2 \in \mathbb{Q}$ st. $a < r_2 < r_1 < b$
$\vdots$
. . . $P(n)$: $r_n \in \mathbb{Q}$ st. $a < r_n < \cdots < r_2 < r_1 < b$.

Proof by Induction

Basis: given $a, b \in \mathbb{R}$ and $a < b$,
then there is a rational $r_1 \in \mathbb{Q}$ st. $a < r_1 < b$. $P(1)$ ✓

Induction: assume $P(n)$ true. show $P(n+1)$ true.
If there is a rational $r_n \in \mathbb{Q}$ st. $a < r_n < b$,
then there is a rational $a < r_{n+1} < r_n < b$.
Since there are ∞ many rational #s btw a and b
$\therefore$ there are ∞ many rationals btw a and b.

4.14

a) Prove $\sup(A+B) = \sup A + \sup B$

By def, $a+b \leq \sup(A+B)$, for $\forall a \in A$, $\forall b \in B$.

$$a \leq \underbrace{\sup(A+B) - b}_{\text{upper bound for } A}$$

$\because$

$\therefore \quad \sup A \leq \sup(A+B) - b$

$$b \leq \underbrace{\sup(A+B) - \sup A}_{\text{upper bound for } B}$$

$\because$

$\therefore \quad \sup B \leq \sup(A+B) - \sup A$

$$\sup B + \sup A \leq \sup(A+B)$$

Want: $\sup(A+B) \leq \sup B + \sup A$

$$a+b \leq \sup(A+B)$$

$\because$ $a \leq \sup A$ for $\forall a \in A$

$b \leq \sup B$ for $\forall b \in B$

$\therefore \quad a+b \leq \underbrace{\sup A + \sup B}_{\text{upper bound for } A \text{ and } B}$

$$\begin{cases} \therefore \ \sup(A+B) \leq \sup A + \sup B \\ \because \ \sup(A+B) \leq \sup A + \sup B \end{cases}$$

$\therefore \quad \sup(A+B) = \sup A + \sup B$

b) Prove $\inf(A+B) = \inf A + \inf B$

By def, $a+b \geq \inf(A+B)$ for $\forall a \in A$, $\forall b \in B$.

$$a \geq \underbrace{\inf(A+B) - b}_{\text{lower bound for } A}$$

$$\inf A \geq \inf(A+B) - b$$

$\because \quad b \geq \underbrace{\inf(A+B) - \inf A}_{\text{lower bound for } B}$

$\therefore \quad \inf B \geq \inf(A+B) - \inf A$

$\therefore \quad \inf A + \inf B \geq \inf(A+B)$

Want: $\inf(A+B) \geq \inf A + \inf B$

By def, $a+b \geq \inf(A+B)$

$\because$ $a \geq \inf A$ for $\forall a \in A$

$b \geq \inf B$ for $\forall b \in B$

$\therefore$ $a+b \geq \underbrace{\inf A + \inf B}_{\text{lower bound for A and B}}$

$\therefore$ $\inf(a+b) \geq \inf A + \inf B$

$\because$ $\inf(a+b) \leq \inf A + \inf B$

$\therefore$ $\inf(a+b) = \inf A + \inf B$

7.5 Limit = ?

a) $\lim S_n = \sqrt{n^2+1} - n$

$$= \lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right)\left(\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right)$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}$$

$$= \lim_{n\to\infty} \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} = \frac{1}{\infty} = \boxed{0}$$

b) $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$

$$= \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)\left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n}$$

$$= \cdots \frac{\frac{n}{n}}{\frac{\sqrt{n^2+n}+n}{n}} = \cdots \frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \cdots \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \cdots \frac{1}{\sqrt{1+0}+1} = \cdots \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

c) $\lim \left(\sqrt{4n^2+n} - 2n \right)$

$$= \lim_{n\to\infty} \left(\sqrt{4n^2+n} - 2n \right) \left(\frac{\sqrt{4n^2+n} + 2n}{\sqrt{4n^2+n} + 2n} \right)$$

$$= \ldots \frac{4n^2+n-4n^2}{\sqrt{4n^2+n} + 2n}$$

$$= \ldots \frac{n}{\sqrt{4n^2+n} + 2n} = \frac{\frac{n}{n}}{\frac{\sqrt{4n^2+n} + 2n}{n}}$$

$$= \ldots \frac{1}{\sqrt{\frac{4n^2+n}{n^2}} + 2}$$

$$= \ldots \frac{1}{\sqrt{4+\frac{1}{n}} + 2}$$

$$= \ldots \frac{1}{\sqrt{4+0} + 2}$$

$$= \ldots \boxed{\frac{1}{4}}$$