HW4

Ross 12.10, 12.12, 14.2, 14.10
Rudin Chp3: 6, 7, 9, 11

12.10 Prove (S_n) is bounded iff $\limsup |S_n| < +\infty$.

"$\subseteq$" if (S_n) is bounded.

then there exists $M \in \mathbb{R}$ st. $|S_n| \leq M$ for $\forall n$.

$\therefore \sup\{|S_n| : n \in \mathbb{N}\} \leq M < +\infty$ $\forall n$.

hence $\limsup |S_n| < +\infty$.

"$\supseteq$" if $\limsup |S_n| < +\infty$.

then there exists $M \in \mathbb{R}$ st. $\limsup |S_n| = M$.

since sequence $A_N = \sup\{|S_n| : n > N\}$ converges to M, take $\varepsilon = 1$

then there exists $N_1 \in \mathbb{N}$ st. $\left| \sup\{|S_n| : n > N_1\} - M \right| < 1$

$\Rightarrow$ $\sup\{|S_n| : n > N_1\} < M+1$

$\Rightarrow$ $|S_n| < M+1$ $\forall n > N$.

take $N = \max\{|S_1|, |S_2|, |S_3|, \dots, |S_N|, M+1\}$.

$\therefore |S_n| \leq N$ $\forall n$.

hence (S_n) is bounded.

$\square$

12.12 $\quad s_n$ sequence of nonnegative #s

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

a) show $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.

(hint: for last inequality, show first that $M > N$ implies $\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1+s_2+\cdots+s_N) + \sup\{s_n : n > N\}$.

$\liminf \sigma_n \le \limsup \sigma_n$ is obvious. So only need to prove $\limsup \sigma_n \le \limsup s_n$, and $\liminf s_n \le \liminf \sigma_n$.

i) WTS $\limsup \sigma_n \le \limsup s_n$

take M and N st. $n > M > N$.

then $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

$= \frac{1}{n}(s_1+s_2+\cdots+s_N + s_{N+1} + s_{N+2} + \cdots + s_n)$

$= \frac{1}{n}(s_1+s_2+\cdots+s_N) + \frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n)$

$\because n > M$

$\therefore \frac{1}{n}(s_1+s_2+\cdots+s_N) < \frac{1}{M}(s_1+s_2+\cdots+s_N)$

$\frac{1}{n}(s_{N+1} + s_{N+2} + \cdots + s_n) \le \sup\{s_n : n > M\}$

$\sigma_n = \frac{1}{n}(s_1+s_2+\cdots+s_N) + \frac{1}{n}(s_{N+1}+s_{N+2}+\cdots+s_n)$

$< \frac{1}{M}(s_1+s_2+\cdots+s_N) + \sup\{s_n : n > M\}$

$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1+s_2+\cdots+s_N) + \sup\{s_n : n > M\}$

$\lim_{M\to\infty} \sup\{\sigma_n : n > M\} \le \lim_{M\to\infty} \frac{1}{M}(s_1+s_2+\cdots+s_N) + \lim_{M\to\infty} \sup\{s_n : n > M\}$

$\limsup \sigma_n \le 0 + \limsup s_n$

$\limsup \sigma_n \le \limsup s_n$.

ii) WTS $\liminf s_n \le \liminf \sigma_n$

let $a_n = -\sigma_n$, $b_n = -s_n$.

$\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$

$a_n = \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$

$\because \limsup s_n = -\liminf(-s_n) \quad \forall s_n$

$\limsup(a_n) \le \limsup(b_n)$

$$-\limsup(a_n) \geq -\limsup(b_n)$$

$$-\limsup(-\sigma_n) \geq -\limsup(-S_n)$$

$$\liminf \sigma_n \geq \liminf S_n$$

$$\liminf S_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup S_n$$

$\square$

b) show if $\lim S_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim S_n$.

by Thm 10.7, if $\lim S_n$ exists $\Rightarrow \liminf S_n = \lim S_n = \limsup S_n$

by a) $\quad \liminf \sigma_n = \limsup \sigma_n$

(Thm 10.7) $\quad \lim \sigma_n = \liminf \sigma_n = \limsup \sigma_n = \lim S_n$

c) Give an example where $\lim \sigma_n$ exists, but $\lim S_n$ does not exist.

let $S_n = (1, 0, 1, 0, \cdots)$

$\lim S_n$ doesn't converge D.N.E.

$\lim \sigma_n \to \frac{1}{2}$

14.2 series converge ?

a) $\sum \frac{n-1}{n^2}$

$b_n = \frac{n}{2}$

$n-1 > \frac{n}{2}$ for $n \geq 10$

$\frac{n-1}{n^2} > \frac{\frac{n}{2}}{n^2}$ $\quad \frac{n}{2} \cdot \frac{1}{n^2} = \frac{1}{2n}$

$\frac{n-1}{n^2} > \frac{1}{2n}$

$P \leq 1$ Div. by P-series test

Div. by Direct Comparsion test (DCT)

b) $\sum (-1)^n$

$\lim_{n\to\infty} (-1)^n \not\to 0$

$\therefore$ diverge by test for Divergence (TFD).

c) $\sum \frac{3n}{n^3}$

$= \sum \frac{3}{n^2} = 3\sum \frac{1}{n^2} \rightarrow P>1$

$\therefore$ converge by P-series

d) $\sum \frac{n^3}{3^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{\frac{(n+1)^3}{3^{(n+1)}}}{\frac{n^3}{3^n}} \right| = \lim \left| \frac{(n+1)^3}{3^{(n+1)}} \cdot \frac{3^n}{n^3} \right|$

$= \lim \left| \frac{(n+1)^3}{3n^3} \right| = \lim_{n\to\infty} \left| \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \right|$

$= \frac{1}{3} < 1$

$\therefore$ converge absolutely by Ratio test

e) $\sum \frac{n^2}{n!}$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+1)\cdot n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{n^2} \right| = \lim \left| \frac{1}{n} + \frac{1}{n^2} \right| = 0$

< 1

$\therefore$ converge absolutely by Ratio test

f) $\sum \frac{1}{n^n}$

$r_n = \sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{1}{n^n}\right|} = \frac{1}{n}$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim \frac{1}{n} = 0$

< 1

$\therefore$ conv. absolutely by Root test

g) $\sum \frac{n}{2^n}$

$= \lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2n} \right|$

$= \lim_{n\to\infty} \left| \frac{1}{2}\left(1+\frac{1}{n}\right) \right| = \frac{1}{2} < 1$

$\therefore$ conv. abso. by Ratio test.

14.10 find a series $\sum a_n$ which diverges by Root test but no info for Ratio test.

div. by root test $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$

no info by Ratio test $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

try $a_n = 2^{(-1)^n n}$.

root test: $|a_n|^{\frac{1}{n}} = |2^{(-1)^n n}|^{\frac{1}{n}} = 2^{(-1)^n} \rightarrow \in \{2, \frac{1}{2})$

$\limsup |a_n|^{\frac{1}{n}} = 2 > 1.$

$\therefore$ div. by root test

· ratio test $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{(-1)^{n+1}}(n+1)}{2^{(-1)^n n}}\right| = 2^{(-1)^{n+1}(2n+1)}$ $\in \{2^{2n+1}, \frac{1}{2^{2n+1}}\}$

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = +\infty$

$\liminf \left|\frac{a_{n+1}}{a_n}\right| = 0$

$\therefore$ no info by ratio test

Rudin Chp3: 6, 7, 9, 11

6. convergence / divergence ?

a) $a_n = \sqrt{n+1} - \sqrt{n}$

partial sum $S_n = \sum_{j=1}^{n} a_n = \sqrt{n+1} - 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n+1} - 1) = \infty$

$\therefore$ Diverges

b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

$= \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n^{3/2}}$ $p = 3/2 > 1$

conv. by P-series

$\therefore$ div. by comparison test

c) $a_n = (\sqrt[n]{n} - 1)^n$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} (\sqrt[n]{n} - 1) = n^{\frac{1}{n}} - 1 = n^0 - 1 = 1 - 1 = 0 < 1$

$\therefore$ conv. by Root test

d) $a_n = \frac{1}{1+z^n}$ $z \in \mathbb{C}$

Case 1: $|z| \leq 1$ $\lim_{n\to\infty} \frac{1}{1+|z|^n} \neq 0$ $\Big\}$ div.

case 2 : $|z| = 1$ $\qquad \lim \frac{1}{1+|z|^n} = \frac{1}{2}$

case 3 : $|z| < 1$ $\qquad \lim \frac{1}{1+|z|^n} = 1$

case 4 : $|z| > 1$ $\qquad \frac{1}{1+|z|^n} \leq \frac{1}{|z|^n}$

conv. by Comparison test

Conv. by fact $\lim \frac{1}{n^p} = 0$ $\forall p > 0$

7. Prove convergence of $\sum a_n$ implies convergence of $\sum \frac{\sqrt{a_n}}{n}$, $a_n \geq 0$.

By AM-GM inequality

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$$2\sqrt{xy} \leq x+y$$

$$2\sqrt{a_n \cdot \frac{1}{n^2}} = 2\frac{\sqrt{a_n}}{n} \leq a_n + \frac{1}{n^2}$$

conv. by comparison test

Conv.

9. Find radius of convergence of Power series.

a) $\sum n^3 z^n$

root test : $\alpha = \limsup |n^3|^{\frac{1}{n}} = \left(\lim\limits_{n\to\infty} n^{\frac{1}{n}}\right)^3 = 1^3$

$R = \frac{1}{\alpha} = 1$

- Power Series :
  - $\sum\limits_{n=0}^{\infty} a_n \cdot x^n \qquad a_n \in \mathbb{R}$
- Thm : let $\alpha = \limsup |a_n|^{\frac{1}{n}}$, let $R = \frac{1}{\alpha}$ ratio of convergence

  then,
  - if $|x| < R$, $\sum a_n x^n$ is absolutely convergent.
  - if $|x| > R$, $\sum a_n x^n$ is divergent.
  - if $|x| = R$, it depends.

b) $\sum \frac{2^n}{n!} z^n$

ratio test : $\alpha = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right|$

$= \lim\limits_{n\to\infty} \left|\frac{2}{n+1}\right| = 0$

$R = \frac{1}{\alpha} = \infty$

c) $\sum \frac{2^n}{n^2} z^n$

root test: $\alpha = \limsup \left| \frac{2^n}{n^2} \right|^{\frac{1}{n}} = 2$

$R = \frac{1}{\alpha} = \frac{1}{2}$

d) $\sum \frac{n^3}{3^n} z^n$

root test: $\alpha = \limsup \left| \frac{n^3}{3^n} \right|^{\frac{1}{n}}$

$R = \frac{1}{\alpha} = 3$

11. given $a_n > 0$, $S_n = a_1 + \cdots + a_n$, $\sum a_n$ div.

a) Prove $\sum \frac{a_n}{1+a_n}$ div.

(i) a_n unbounded

then $\frac{a_n}{1+a_n}$ is also unbounded

$\therefore \sum \frac{a_n}{1+a_n}$ div.

(ii) a_n bounded ie. $a_n \leq M$

then $\sum \frac{a_n}{1+a_n} \geq \sum \frac{a_n}{1+M} = \frac{1}{M+1} \sum a_n$.

div.

b) Prove $\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \geq 1 - \frac{S_N}{S_{N+k}}$

and deduce $\sum \frac{a_n}{S_n}$ div.

$\because S_n \uparrow$. $a_n > 0$, $S_{n+1} > S_n$ $\forall n$.

$$\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1}}{S_{N+k}} + \cdots + \frac{a_{N+k}}{S_{N+k}}$$

$$= \frac{a_{N+1} + \cdots + a_{N+k}}{S_{N+k}}$$

$$= \frac{S_{N+k} - S_N}{S_{N+k}}$$

$$= 1 - \frac{S_N}{S_{N+k}}$$

assume $\frac{a_n}{S_n}$ conv.

then $0 < \varepsilon < 1$, $\exists N \in \mathbb{N}$ st.

$$\varepsilon > \frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} > 1 - \frac{S_N}{S_{N+k}} \quad \forall k.$$

$\because \sum a_n$ div. $\{S_n\}$ unbounded $\therefore S_n \to \infty$.

let $k \to \infty$,

we get $\varepsilon \geq 1$, which is a contradictin w/ $\varepsilon < 1$.

Hence $\sum \frac{a_n}{S_n}$ diverges.

c) Prove $\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$

deduce $\sum \frac{a_n}{S_n^2}$ conv.

$\because S_n > S_{n-1} \quad \forall n \geq 1$

$$\frac{a_n}{S_n^2} \leq \frac{a_n}{S_{n-1} S_n} = \frac{S_n - S_{n-1}}{S_{n-1} S_n} = \frac{1}{S_{n-1}} - \frac{1}{S_n}$$

$$\sum_{n=0}^{N} \frac{a_n}{S_n^2} \leq 1 + \sum_{n=1}^{N} \left(\frac{1}{S_{n-1}} - \frac{1}{S_n} \right) = 1 + \frac{1}{a_0} - \frac{1}{S_N} \leq 1 + \frac{1}{a_0}.$$

then all partial sums $\sum_{n=0}^{N} \frac{a_n}{S_n^2}$ are bounded by $1 + \frac{1}{a_0}$.

Hence $\sum_{n=0}^{N} \frac{a_n}{S_n^2}$ conv.

d) $\sum \frac{a_n}{1+na_n}$, $\sum \frac{a_n}{1+n^2 a_n}$?

(i) $\frac{a_n}{1+n^2 a_n} \leq \frac{a_n}{n^2 a_n} = \frac{1}{n^2}$

$\uparrow$ conv. by comparison test $\qquad \uparrow$ conv. by p-series

(ii) $a_n = \frac{1}{n}$

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1}{2n} \text{ div.}$$

(iii) $\begin{cases} a_n = 1, & n = 2^k, \ \exists k \in \mathbb{N} \\ a_n = 2^{-n} & \text{else} \end{cases}$

$\sum a_n \geq \sum_{k=0}^{\infty} 1$ diverges

but $\sum \frac{a_n}{1+na_n} \leq \sum_{k=0}^{\infty} \frac{1}{1+2^k} + \sum_{n=1}^{\infty} \frac{1}{2^n+n}$ conv.

$\therefore \frac{a_n}{1+na_n}$ can be converge or diverge.