Math 104 HW 4

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8/27/2020

Ross 12.10

If it is bounded $|s_n| \leq M$ for $M \in \mathbb{R}$ then $\sup\limits_{n \geq N} |s_n| \leq M$ for all N and therefore $\limsup s_n \leq M < \infty$. Conversely suppose $\limsup |s_n| < \infty$. Then for any $\varepsilon > 0$, there must exist N such that for all $n > N$ $|s_n| \leq \limsup |s_n| + \varepsilon$. Then $|s_n| \leq \max(\max\limits_{0 \leq n \leq N} |s_n|, \limsup |s_n| + \varepsilon)$, as the finite beginning of the sequence must be bounded and the tail must also be bounded due to the limsup condition.

12.12

(a) With $n > M > N$ then we have:

$$\begin{aligned}
\sigma_n = \frac{1}{n} \sum_{i=1}^{n} s_i &= \frac{\sum_{i=1}^{N} s_i}{n} + \frac{\sum_{i=N+1}^{n} s_i}{n} \\
&\leq \frac{\sum_{i=1}^{N} s_i}{M} + \frac{\sum_{i=N+1}^{n} s_i}{n} \\
&\leq \frac{\sum_{i=1}^{N} s_i}{M} + \frac{n-N}{n} \sup_{n \geq N} \sup\{s_n : n > N\} \\
&\leq \frac{\sum_{i=1}^{N} s_i}{M} + \sup_{n \geq N} \sup\{s_n : n > N\}
\end{aligned}$$

As this is an upper bound it must hold for the supremum:

$$\sup\{\sigma_n : n \geq M\} \leq \frac{\sum_{i=1}^{N} s_i}{M} + \sup\{s_n : n > N\}$$

Then by taking the limit as $M \to \infty$ (note that $M > N$ so N can remain constant) and then the limit as $N \to \infty$

$$\begin{aligned}
\limsup \sigma_n &\leq \sup\{s_n : n > N\} \\
\limsup \sigma_n &\leq \limsup s_n
\end{aligned}$$

The middle inequality follows by definition and taking limits as $\inf\{\sigma_n : n > N\} \leq \sup\{\sigma_n : n > N\}$. The leftmost inequality follows by negating the lim sup inequality.

(b) Let $s_n = (-1)^n$. Then $\lim \sigma_n = 0$.

14.2

(a)

$$\begin{aligned} \frac{n-1}{n^2} &\geq \frac{1}{n+2} \\ n^2+n-2=(n-1)(n+2) &\geq n^2 \\ n &\geq 2 \\ a_n &= \begin{cases} 0 & n=1 \\ \dfrac{1}{n+2} & n \geq 2 \end{cases} \end{aligned}$$

Then $a_n \leq \dfrac{n-1}{n^2}$ but $\sum a_n = \infty$ as it is the harmonic sum (or at least it's tail)

(b) It does not, the $\limsup = 1$ and $\liminf = -1$

(c) It does, it is $3 \sum \dfrac{1}{n^2}$ which follows by linearity of limits.

(d) It will converge by the root test

$$\limsup \left| \left(\frac{n^3}{3^n} \right)^{1/n} \right| = \limsup \left| \frac{n^{3/n}}{3} \right| = \frac{1}{3}$$

(e) It will diverge by the ratio test

$$\begin{aligned} \limsup \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} &= \limsup \frac{1}{n+1} \frac{n^2+2n+n}{n^2} \\ &= \infty \cdot 1 > 0 \end{aligned}$$

(f) For $n > 2$, then $0 < \dfrac{1}{n^n} \leq \dfrac{1}{n^2}$ so it converges.

(g) By the same argument as d it will converge.

Ross 14.10

Let

$$a_{n+1} = \begin{cases} \frac{1}{2} a_n & n \text{ odd} \\ 3a_n & n \text{ even} \end{cases}$$

And treat $a_0 = 1$. Then

$$\lim a_n^{1/n} = \sqrt{3/2} > 1$$

While $\liminf a_{n+1}/a_n = 1/2 < 1 < \limsup a_n = 3$.

Rudin 3.6

(a)

$$\sqrt{n+1} - \sqrt{n} \geq \frac{1}{n}$$
$$2n + 1 - 2\sqrt{n^2 + n} \geq \frac{1}{n^2}$$
$$2n^3 + n^2 - 2n^2\sqrt{n^2 + n} \geq 1$$

Which numerically holds true for $n \geq 5$ and therefore the series diverges. Oh, I just realized the terms of the series cancel out so every partial sum is of the form $\sum_{i=1}^{N} a_i = \sqrt{N+1} - 1 \to \infty$

(b) The same cancelling argument holds but it gives partial sums of the form $\frac{\sqrt{N+1}}{N} \to 0$.

(c)

$$\begin{aligned}\limsup |a_n^{1/n}| &= \limsup |n^{1/n} - 1| \\ &= 0\end{aligned}$$

So it converges by the root test

(d) By the sanity test for convergence, I need $|1 + z^n| \to \infty$ so $|z| > 1$.

$$\begin{aligned}\lim |\frac{1 + z^n}{1 + z^{n+1}}| &= \lim |\frac{1 + z^{-n}}{z + z^{-n}}| \\ &= \frac{1}{|z|}\end{aligned}$$

So the series converges absolutely for $|z| > 1$ and diverges for $|z| < 1$. For $|z| = 1$, the series will diverge as the terms cannot change in sign because the denominator will always have a non-negative real part.

Rudin 3.7

By an arcane inequality:

$$\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$$
$$a_n - 2\frac{\sqrt{a_n}}{n} + \frac{1}{n^2} \geq 0$$
$$\frac{1}{2}\left(a_n + \frac{1}{n^2}\right) \geq a_n \geq 0$$

Therefore the series converges.

Rudin 3.9

1.

$$\lim |\frac{(n+1)^3 z^{n+1}}{nz^n}| = \lim |z| < 1$$

If $|z| = 1$, then the n^3 term will prevent convergence as $|a_n| = |n^3||z|^n$. So the radius is $|z| < 1$

2.

$$\lim \left| \frac{2^{n+1} z^{n+1} n!}{(n+1)! 2^n z^n} \right| = \lim \left| \frac{2z}{n+1} \right| = 0$$

So it will always converge.

3.

$$\lim \left| \frac{2^{n+1} z^{n+1} n^2}{2^n z^n (n+1)^2} \right| = |2z|$$

So it will converge for $|z| < \dfrac{1}{2}$. If $|z| = 1/2$ then we get $|a_n| = \dfrac{1}{n^2}$ so it will absolutely converge. So the radius is $|z| \leq 1/2$

4.

$$\lim \left| \frac{(n+1)^3 z^{n+1} 3^n}{n^3 z^n 3^{n+1}} \right| = \lim \left| \frac{z}{3} \right|$$

So $|z| \leq \dfrac{1}{3}$ is a sufficient condition for convergence. When $|z| = 1/3$ then we end up with $|a_n| = n^3$ so the series will diverge.

Ross 3.11

(a) By another arcane inequality:

$$\begin{aligned} \frac{a_i}{1+a_i} &> a_i + f \\ a_i &> a_i + a_i^2 + f + f a_i \\ -f(1+a_i) &> a_i^2 \\ a_i/2 + a_i^2/2 &> a_i^2 \\ a_i &> a_i/2 + a_i^2/2 \\ a_i/2 &> a_i^2/2 \end{aligned}$$

Then $\min(a_i/2, 1) \leq \dfrac{a_i}{1+a_u}$. Then if $\sum a_i = \infty$, then I claim $\sum \min(a_i/2, 1) = \infty$ as well. If infinitely many terms $a_i > 1$ then clearly $\sum \min(a_i/2, 1) = \infty$ anyways as infinitely many 1's will still cause divergence. If only finitely many $a_i > 1$, then summing over a subsequence of terms $a_{i_k} < 1$ will diverge as well and therefore so will $\sum \min(a_i/2, 1)$

(b)

$$\begin{aligned} \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &\geq \frac{s_{N+k} - s_N}{s_{N+k}} \\ &\geq 1 - \frac{s_N}{s_{N+k}} \end{aligned}$$

Therefore the tails are sufficiently heavy to diverge. For any N you can constuct a tail tail by choosing finite $N+k$ such that $s_{N+k} \geq 2s_N$ (which exists because the series diverges):

$$\sum_{i=N}^{N+k} \frac{a_i}{s_i} \geq \frac{1}{2}$$

Then you can repeat the process with $N+k+1$ as the new starting point for constructing another heavy tail with sum $\geq 1/2$.

(c)

$$\begin{aligned} \frac{1}{s_{n-1}} - \frac{1}{s_n} &= \frac{s_n - s_{n-1}}{s_n s_{n-1}} \\ &= \frac{a_n}{s_n s_{n-1}} \\ &\geq \frac{a_n}{s_n^2} \end{aligned}$$

I have no intuition on this one. Oh, it says converges. That makes things much easier.

$$03_{n\to\infty} a_1 s_i^2 + \tfrac{1}{s_1} < \infty \sum_{i=1}^{n} \tfrac{a_i}{s_i^n} \leq \sum_{i=1}^{n} \tfrac{1}{s_{i-1}} - \tfrac{1}{s_i} = \tfrac{a_1}{s_i^2} + \tfrac{1}{s_1} - \tfrac{1}{s_n}$$

$03_{n\to\infty} a_1 s_i^2 + \tfrac{1}{s_1} < \infty$

So series converges by comparison

(d) That will depend on the series. For instance with $a_n = 1$ then $\sum \frac{1}{1+n} = \infty$ and with $a_n = \frac{1}{n^k}$ then $\sum \frac{1}{n^k(1+\frac{n}{n^k})} = \sum \frac{1}{n^k + n}$ which will converge for $k > 1$.

For the second summation it will always converge as $\frac{a_n}{1+n^2 a_n} = \frac{1}{\frac{1}{a_n} + n^2}$. If you allowed $a_n < 0$ then you could get it to not converge with something like $a_n = -\frac{1}{2n^2}$, but as $a_n > 0$ then $\frac{a_n}{1+n^2 a_n} \leq \frac{1}{n^2}$ so the series converges by comparison.