HW5 Ross 13.3, 13.5, 13.7

13.3 B = set of all bounded sequences $x = (x_1, x_2, \cdots)$, and define

$d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots\}$.

a) show d is a metric for B.

① let $x \in B$.

given $d(x,y) = \sup\{|x_j - y_j| : j = 1, 2, \cdots\}$ note: $\forall j = \{1, 2, \cdots\}$

$d(x,x) = \sup\{|x_j - x_j|\} = 0$ ✓

let $x \neq y \in B$.

then $\exists j$ st. $x_j \neq y_j$, and so $|x_j - y_j| > 0$,

$d(x,y) = \sup|x_j - y_j| > 0$ ✓

② let $x, y \in B$.

$d(x,y) = \sup|x_j - y_j|$

$= \sup|y_j - x_j|$

$= d(y,x)$ ✓

③ let $x, y, z \in B$.

$d(x,y) = \sup|x_j - y_j|$

$d(x,z) = \sup|x_j - z_j|$

$= \sup|x_j - y_j + y_j - z_j|$

triangle ineq.

$\leq \sup|x_j - y_j| + \sup|y_j - z_j|$

$\leq d(x,y) + d(y,z)$ ✓

Hence, function d is matric for B. $\square$

b) Does $d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for B?

take $x = (1, 1, 1, \dots)$ and $y = (0, 0, 0, \dots)$.

$x, y \in B$

$$d^*(x,y) = \sum_{j=1}^{\infty} |x_j - y_j|$$

$$= \sum_{j=1}^{\infty} |1 - 0|$$

$$= 1$$

$$= \infty$$

$\because$ distance btw x and y $= \infty$

$\therefore$ No. $d^*(x,y)$ is <u>not</u> a metric for B. $\square$

13.5 a) verify DeMorgan's Laws for sets

$$\bigcap\{S\setminus U : U\in\mathcal{U}\} = S\setminus\bigcup\{U : U\in\mathcal{U}\}.$$

$$\begin{aligned} S\setminus\bigcup\{U : U\in\mathcal{U}\} &= S\cap\{\bigcup U : U\in\mathcal{U}\}^c \\ &= S\cap\{\bigcap U^c : U\in\mathcal{U}\} \\ &= \bigcap\{S\cap U^c : U\in\mathcal{U}\} \\ &= \bigcap\{S\setminus U : U\in\mathcal{U}\} \end{aligned}$$

$\square$

b) Show intersection of any collection of closed sets is a closed set.

let S an open set.

take arbitary elem $U\in S$.

Then, $\bigcup_{U\in S} U = A$ is an open set.

By DeMorgan, $\left(\bigcup_{U\in S} U\right)^c = \bigcap_{U\in S} U^c = B$. B is a closed set ($\because$ complement of A)

B is an infinite intersection of closed set U^c.

Hence, infinite intersection of closed sets is a closed set.

$\square$

13.7 Show every open set in $\mathbb{R}$ is disjoint union of a finite or infinite sequence of open intervals.

- let $U \subset \mathbb{R}$ be an open set.

  let $x \in U$.

  by def of open set, there exist an open interval I_x st. $x \in I_x \subseteq U$.

  $x \in U \Rightarrow x \in I_x \Rightarrow x \in \bigcup_{x \in U} I_x$

  $\therefore \quad U \subseteq \bigcup_{x \in U} I_x$.

  also, Since $\forall x \in U, I_x \in U$,

  $\therefore \quad \bigcup_{x \in U} I_x \subseteq U$.

  so, $\underbrace{U = \bigcup_{x \in G} I_x}$ (ie. open sets = union of open intervals)

- let $I_x^* = (\alpha^*, \beta^*)$ be the largest interval, where

  $\alpha^* = \inf\{\alpha < x : (\alpha, x) \subseteq U\}$

  $\beta^* = \sup\{\beta > x : (x, \beta) \subseteq U\}$

  clearly, $\alpha^* < x < \beta^*$ $\quad \therefore x \in I_x^*$.

  By def of infimum and supremum,

  for every $\varepsilon > 0$, $\alpha^* + \varepsilon \in (\alpha, x) \subseteq U$, and $\beta^* - \varepsilon \in (x, \beta) \subseteq U$ for some α and β.

  ie. for every $\varepsilon > 0$, $\alpha^* + \varepsilon \in U$ and $\beta^* - \varepsilon \in U$.

  $\therefore \quad I_x^* = (\alpha^*, \beta^*) \subseteq U$.

  So, $\forall x \in U, x \in I_x^* \subseteq U \quad \Rightarrow \underbrace{U = \bigcup_{x \in U} I_x^*}$

- claim: $\{I_x^*\}$ is disjoint

  ie. for $x, y \in U$ either $I_x^* = I_y^*$ or $I_x^* \cap I_y^* = \emptyset$.

  suppose, $I_x^* \neq I_y^*$ and $I_x^* \cap I_y^* \neq \emptyset$.

  then $I_x^* \cup I_y^*$ will be a larger interval than I_x^* containing x.

  contradiction w/ I_x^* is the max interval.

  So, I_x^* is $\underbrace{\text{disjoint}}$ and $U = \bigcup_{x \in U} I_x^*$.

- suppose, $\{I_x^*\}$ written as $\{I_x^*, I_y^*, I_z^*, \cdots\}$

  Then, there exist distinct rational numbers $x_q, y_q, z_q, \cdots$

  st. $x_q \in I_x^*$, $y_q \in I_y^*$, $z_q \in I_z^*$, etc.

  Since the set of rational numbers is countable,

  so, $\{I_x^*\}$ of open intervals is countable.

  Hence, U is a $\underbrace{\text{countable}}$ union of disjoint open intervals I_x^*.

  $\downarrow$

  finite and infinitely countable

□

4. given (X, d) a metric space, and S a subset of X, we defined the closure of S
$\bar{S} = \{p \in X \mid$ there exist a seq (P_n) in S st. $P_n \to P.\}$
Prove taking closure again won't make it any bigger.
i.e. if $S_1 = \bar{S}$, and $S_2 = \bar{S_1}$, then $S_1 = S_2$.

suppose $x \in \bar{\bar{S}}$. let U be an open set containing x. we want to show $U \cap S \neq \emptyset$.
we know $U \cap \bar{S} \neq \emptyset$, so there exists a seq$(P_n) = y \in \bar{S}$ st. $P_n \to P$, ie. $y \in U$.
But since U is an open set contains y and $y \in \bar{S}$, then $U \cap S \neq \emptyset$.
so, $x \in \bar{S}$. hence, by def of subset, $\bar{\bar{S}} \subseteq \bar{S}$. (taking closure again won't get any bigger than $\bar{S}$).

$\square$

5. Prove $\bar{S}$ is the intersection of all closed subsets in X that contains S. (may assume $\bar{S}$ is closed.)

take $a \in \bar{S}$. let X be any closed set st. $S \subseteq X$.
suppose $a \notin X$. then $a \in a \setminus X$, and $a \setminus X$ is open, so $(a \setminus X) \cap S \neq \emptyset$.
But, $S \subseteq X$. so, $S \cap (a \setminus X) = \emptyset$. contradiction shows that $a \in X$.
$\because$ X is arbitary closed set contaiing S.
$\therefore$ $\bar{S} \subseteq \cap \{a \subseteq X : S \subseteq X$ and X is closed$\}$.
also, observe $\bar{S}$ is one of the closed sets containing S,
so, if $a \in \cap \{a \subseteq X : S \subseteq X$ and X is closed$\}$,
then, automatically $x \in \bar{S}$.
Hence, $\cap \{a \subseteq X : S \subseteq X$ and X is closed$\} \subseteq \bar{S}$.

$\square$