Math 104 HW 5

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Ross 13.3

(a)

$$\sup_{j\geq 1}|x_j - y_j| = 0 \implies x_j = y_j \quad \forall j$$

Conversely if $x_k \neq y_k$ for some k then $\sup_{j\geq 1}|x_j - y_j| \geq |x_k - y_k| > 0$. It is symmetric as $|x_j - y_j| = |y_j - x_j|$. The triangle inequality is inherited from that of $|\cdot|$:

$$|x_i - z_i| = |x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$$
$$\sup_{i\geq 1}|x_i - z_i| \leq \sup_{i\geq 1}(|x_i - y_i| + |y_i - z_i|) \leq \sup_{i\geq 1}|x_i - y_i| + \sup_{i\geq 1}|y_i - z_i|$$

The assumption of bounded sequences ensures the supremum exists.

(b) There can be issues where $\sum_{j\geq 1}|x_j - y_j|$ is divergent, but if we include ∞ as a valid value then it should work.

$$\sum_{j\geq 1}|x_j - y_j| = 0 \implies x_j = y_j \quad \forall j$$

Conversely if $x_k \neq y_k$ for some k then $\sum_{j\geq 1}|x_j - y_j| \geq |x_k - y_k| > 0$. It is symmetric as $|x_j - y_j| = |y_j - x_j|$ and therefore all partial sums are equal. The triangle inequality is inherited from that of $|\cdot|$:

$$|x_i - z_i| = |x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$$
$$\sum_{i\geq 1}|x_i - z_i| \leq \sum_{i\geq 1}(|x_i - y_i| + |y_i - z_i|) \leq \sum_{i\geq 1}|x_i - y_i| + \sum_{i\geq 1}|y_i - z_i|$$

This works for the infinite series, as the inequality holds for every partial sum. There is an issue where you could have $\infty < \infty$ though.

13.5

(a) Let $x \in \bigcap_{U \in \mathcal{U}} S \setminus U$. Then note that $S \setminus U = S \cap U^C$.

$$\begin{aligned} & x \in S \setminus U \quad \forall U \in \mathcal{U} \\ \implies & x \in S \\ \implies & x \in U^C \quad \forall U \in \mathcal{U} \\ \implies & x \in S \cap \bigcap_{U \in \mathcal{U}} U^C = S \setminus \bigcup_{U \in \mathcal{U}} U \end{aligned}$$

Conversely let $x \in S \setminus \bigcup_{U \in \mathcal{U}} U = S \cap \bigcap_{U \in \mathcal{U}} U^C$. Then

$$\begin{aligned} & x \in S \\ & x \in U^C \quad \forall U \in \mathcal{U} \\ & x \in S \cap U^C \quad \forall U \in \mathcal{U} \\ \implies & x \in \bigcap_{U \in \mathcal{U}} S \setminus U \end{aligned}$$

(b) To apply the just shown de-morgan's law consider each closed set: $R = S \setminus U^C$. Then for any $\mathcal{R}$ collection of closed sets we can define $\mathcal{U} = \{R^C : R \in \mathcal{R}\}$. Then $\bigcap_{R \in \mathcal{R}} R = \bigcap_{U \in \mathcal{U}} S \setminus U = S \setminus \bigcup_{U \in \mathcal{U}} U$. Then $\bigcup_{U \in \mathcal{U}} U$ is an open set as the union of open sets and therefore it's complement $\bigcap_{R \in \mathcal{R}} R$ is closed.

13.7

Let $U \subseteq R$ be an open set. Then for any $u \in U$ there is a corresponding $r_u > 0$ such that $(u - r_u, u + r_u) \subseteq U$. Then $\hat{U} = \bigcup_{u \in U} (u - r_u, u + r_u)$ is an open set and I claim it is equal to U. Clearly $U \subseteq \hat{U}$, so let $\hat{u} \in \hat{U}$. Then for some $u \in U$ and $r_u > 0$, $\hat{u} \in (u - r_u, u + r_u) \subseteq U$. So $\hat{U} = U$.

Repeated Closure

First, note that $S \subseteq \bar{S} = S_1$ as for any $p \in S$, the constant sequence $p_n = p \to p$. Then let $S_2 = \bar{S_1}$ and suppose $x \in S_2 \setminus S_1$. Then for some $\varepsilon > 0$, $d(s, x) > \varepsilon$ for all $s \in S$. Otherwise there would be a convergent sequence $s_n \in S$ where $s_n \to x$.
But there must be a sequence $\bar{s}_n \in S_1$ where $\bar{s}_n \to x$, and then for some N $d(\bar{s}_N, x) < \varepsilon/2$. And as $\bar{s}_N \in S_1$ there must be a sequence $s_n \to \bar{s}_N$ and some M such that $d(\bar{s}_N, s_M) < \varepsilon/2$. Then by the triangle inequality $d(x, s_M) \leq d(\bar{s}_N, s_M) + d(\bar{s}_N, s_M) < \varepsilon$, contradicting our assumption $x \notin S_1$.

Intersection of all containing closed sets

Let $\mathcal{R} = \{R \subseteq X : R \text{ closed}, S \subseteq R\}$ be the set of all closed sets containing $S \subseteq X$. Then note that $S \subseteq \bar{S}$ as before so $\bar{S} \in \mathcal{R}$ and therefore $\bigcap_{R \in \mathcal{R}} R \subseteq \bar{S}$.

Conversely let $x \in \bar{S}$ and $R \in \mathcal{R}$ be any closed set containing S. Then there exists $p_n \in S \subseteq R$ such that $p_n \to x$. Then $p_n \in R$ and $p_n \to x$, and as R is closed, $x \in R$. Therefore $\bar{S} \subseteq \bigcap_{R \in \mathcal{R}} R$, as the argument holds for any R