Math 104 HW 6

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Product is Compact

Let $(x_n, y_n) \in X \times Y$ be sequentially compact. Then x_n is a sequence in X and therefore there exists a convergent subsequence x_{n_m}. Then y_{n_m} is a subsequence in Y and therefore there exists a convergent subsequence $y_{n_{m_k}}$. Then $(x_{n_{m_k}}, y_{n_{m_k}})$ is a convergent sequence in $X \times Y$

Unusual decimal expansion

It is not countable for the same reason binary sequences are not countable via a diagonalization argument. As given any enumeration a_{n_k} you can construct a number $b_n \neq a_{n_n}$ by flipping 4 and 7. For any $a \in [0,1]$, the binary expansion $a = \frac{a_n}{2^n}$, $a_n \in \{0,1\}$ has $f\left(\sum_{n=1}^{\infty} \frac{h^{-1}(a_n)}{10^n}\right) = a$ so it is surjective. It is injective as for any $x, y \in E$ we have some first digit $x_n \neq y_n$. Without loss of generality suppose $x_n = 4, y_n = 7$. Then $f(x) \neq f(y)$. The set is clearly bounded, so by the Heine borel theorem I only need to show the set is closed to prove it is bounded. Then consider the complement $x \in \mathbb{R} \setminus E$. For points $x > 1$ $B_{x-1}(x) \subset R \setminus E$ and similarly for $x < 0$ $B_x(x) \subset R \setminus E$. So consider the case $x \in [0,1] \setminus E$ so $x = \sum_{n=1}^{\infty} \frac{x_n}{10^n}$ with $x_N \notin \{4,7\}$ the first digit not in $\{4,7\}$. Then certainly no point $y \in E$ could have $d(x,y) < \frac{1}{10^{N+1}}$ so $B_{10^{-N-1}}(x) \in \mathbb{R} \setminus E$.

$$
\begin{aligned}
x - y &= \frac{x_N - y_N}{10^N} + \sum_{n=1}^{N-1} \frac{x_n - y_n}{10^n} + \sum_{n=N+1}^{\infty} \frac{x_n - y_n}{10^n} \\
&= \frac{x_N - y_N}{10^N} + \sum_{n=N+1}^{\infty} \frac{x_n - y_n}{10^n}
\end{aligned}
$$

$$
\left| \sum_{n=N+1}^{\infty} \frac{x_n - y_n}{10^n} \right| \leq \sum_{n=N+1}^{\infty} \frac{7}{10^n} < \frac{8}{10^{n+1}}
$$

$$
\frac{x_N - y_N}{10^N} > \frac{1}{10^n}
$$

Closures

$$
\begin{aligned}
A_i &= \left(0, 1 - \frac{1}{2^i}\right) \\
B &= (0, 1) \\
\overline{B} &= [0, 1] \\
\overline{A_i} &= \left[0, 1 - \frac{1}{2^i}\right] \\
\bigcup_i \overline{A_i} &= [0, 1]
\end{aligned}
$$

So the inclusion can be strict

Countable union of closed intervals

This is wrong because even though every open set in $\mathbb{R}$ is a countable union of open intervals, the complement is an intersection of closed intervals. The union of closed intervals cannot be used to make sets which have an uncountable number of isolated points such as the cantor set or the set E in problem 2. This can be seen most clearly with the Cantor set, as pugh shows the set is nowhere dense and contains no interval (a, b) and therefore it contains no intervals $[a, b]$ unless $a = b$, but the cantor set is uncountable so you cannot form it as a union of intervals $[a, a]$.