Math 104 HW 8

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8/27/2020

Question 1

For any $\varepsilon > 0$

$$\begin{aligned} f_n(x) - (1/2) &= \frac{n + \sin x}{2n + \cos(n^2 x)} - \frac{1}{2} \\ &= \frac{2n + 2\sin x}{2(2n + \cos(n^2 x))} - \frac{2n + \cos(n^2 x)}{2(2n + \cos(n^2 x))} \\ &= \frac{2\sin x - \cos(n^2 x)}{2(2n + \cos(n^2 x))} \\ |f_n(x) - (1/2)| &\leq \frac{|2| + |1|}{4|n| + 2} \end{aligned}$$

So for sufficiently large n, $|f_n(x) - 1/2| \leq \varepsilon$ for all x.

Question 2

Let $f_N(x) = \sum_{n=1}^{N} a_n x^n$. Then for any $\varepsilon > 0$, I claim the cauchy condition for uniform convergence holds.

$$\begin{aligned} |f_n(x) - f_{m-1}(x)| &= |a_m x^m + \cdots + a_n x^n| \\ &\leq |a_m + \cdots + a_n| \end{aligned}$$

And the summation of $|a_m + \cdots + a_n| < \varepsilon$ for sufficiently large m, n because $\sum |a_n| < \infty$ and therefore the cauchy condition for the tail sum holds. Therefore $f(x)$ is continuous as the uniform limit of $f_N(x)$, which are a family of continuous function. Then because $\sum_{n=1}^{\infty} |n^{-2}| < \infty$, so the given function is continious.

Question 3

For any $0 < a < 1$, then for $x \in [-a, a]$ I have $|x^n| \leq a^n$ and so by the weierstrass M-test, $f(x) = \sum_n x^n$ converges uniformly on E so it is continuous on $[-a, a]$. Then for any $x \in (-1, 1)$, there exists $[-a, a] \ni x$ so f is continuous at x.

The convergence to is not uniform. Note that this series is a geometric one so $\sum_n x^n \to \frac{1}{1-x}$. Let $\varepsilon > 0$ and then

$$\begin{aligned}\frac{1}{1-x} - \sum_{i=0}^{n} x^i &= \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} \\ &= \frac{x^{n+1}}{1-x}\end{aligned}$$

And as $x \to \pm 1$, we have that $\left|\frac{x^{n+1}}{1-x}\right| \to \infty$ for any n and therefore there must exist x sufficiently close to ± 1 and $y = 0$ such that $|f_n(x) - f(x)| > \varepsilon$.