Math 104 HW 8

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Ross p257

One such function, using the same function as shown on pg 257 of Ross as a building block:

$$g(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-1/x) & x > 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{g(x)}{g(x) + g(1-x)} & x \in (0,1) \\ 1 & x \geq 1 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = \frac{0}{0 + g(1)} = 0$$

$$\lim_{x \to 1^+} f(x) = \frac{g(1)}{g(1) + 0} = 1$$

Then f is smooth as it is made of the addition and division of smooth functions g for $x \in (0,1)$. Using the quotient rule $f'(x) = \dfrac{g'(x)\,(g(x) + g(1-x)) - g(x)\,(g'(x) - g'(1-x))}{(g(x) + g(1-x))^2}$ and so on for higher derivatives of f, which will be expressable in terms of g and it's derivatives. Showing $f(x) \in [0,1]$ is kind of hard, but it's very clear from a graphing calculator like desmos.

Rudin Ch 4 Ex 4

Let $f(x_1) \in f(X)$ for any $x_1 \in X$ and $\varepsilon > 0$. Then by continuity of f, for some $\delta > 0$, then $d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$. Then for any $f(x_1) \in f(E)$ (ie $x_1\ inE$), there must be $x_2 \in E$ such that $d_X(x_1, x_2) < \delta$ by density of E and therefore $d_Y(f(x_1), f(x_2)) < \varepsilon$. So $f(Y)$ is dense in $f(X)$.

Suppose $f(p) = g(p)$ for all $p \in E$. Then for any $p \in X$, there must be a sequence of points $p_n \in E$ such that $p_n \to p$ by density of E. Then by continuity of f, g, we must have $f(p) = \lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} g(p_n) = g(p)$

Oh, chapter 5 would make a lot more sense.

Rudin ch 5 ex 4

$$\begin{aligned}
f(x) &= \sum_{i=0}^{n} C_i x^i \\
F(x) &= \sum_{i=0}^{n} \frac{C_i x^{i+1}}{i+1} \\
F'(x) &= f(x) \\
F(1) &= C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0 \\
F(0) &= 0
\end{aligned}$$

Then by Rolle's theorem applied to F, we have $f(c) = 0$ for some $c \in (0, 1)$

Rudin ch 5 ex 8

Let $\varepsilon > 0$ and f' be continuous on $[a, b]$. Then $f' : [a, b] \to \mathbb{R}$ is uniformly continuous as a continuous function on a compact set. Then by uniform continuity, for some $\delta > 0$, $f'(x, y) < \varepsilon$ for any $|x - y| < \delta$, $x, y \in [a, b]$. Then suppose $a \le x \le t \le b$ and $|t - x| < \delta$. Then by the mean value theorem, for some $c \in [x, t]$

$$\frac{f(t) - f(x)}{t - x} = f'(c)$$

Then $|c - x| \le |x - x| + |x - t| < \delta$ as well, so $|f'(c) - f'(t)| < \varepsilon$, as desired.

Rudin ch 5, Ex 18

$$\begin{aligned}
f(t) - f(\beta) &= (t - \beta)Q(t) \\
f'(t) &= \frac{d}{dt}\left((t - \beta)Q(t)\right) \\
&= Q(t) + Q'(t)(t - \beta) \\
f''(t) &= Q'(t) + Q'(t) + (t - \beta)Q''(t) \\
&= 2Q'(t) + Q''(t)(t - \beta) \\
f^{(n)}(t) &= nQ^{(n-1)}(t) + (t - \beta)Q^{(n)}(t) \\
f^{(n)}(\alpha) &= nQ^{(n-1)}(\alpha) + (\alpha - \beta)Q^{(n)}(\alpha) \\
Q^{(n)}(\alpha) &= \frac{f^{(n+1)}(\alpha)}{n+1} + \frac{(\beta - \alpha)Q^{(n+1)}(\alpha)}{n+1} \\
Q^{(n)}(\alpha) &= \frac{f^{(n+1)}(\alpha)}{n+1} + \frac{(\beta - \alpha)\left(\frac{f^{(n+2)}(\alpha)}{n+2} + \frac{(\beta-\alpha)Q^{(n+2)}(\alpha)}{n+2}\right)}{n+1} \\
&= \frac{f^{(n+1)}(\alpha)}{n+1} + (\beta - \alpha)\frac{f^{(n+2)}(\alpha)}{(n+1)(n+2)} + (\beta - \alpha)^2 \frac{Q^{(n+2)}(\alpha)}{(n+2)(n+1)} \\
&= \sum_{k=0}^{M} (\beta - \alpha)^k \frac{f^{(n+1+k)}(\alpha)}{\prod_{m=0}^{k}(n+1+m)} + (\beta - \alpha)^{M+1} \frac{Q^{(n+1+M)}(\alpha)}{\prod_{m=0}^{M}(n+1+m)}
\end{aligned}$$

Then by applying the just shown formula for $Q(t) = Q^{(0)}(t)$ we get the desired result:

$$
\begin{aligned}
Q(t) &= \sum_{k=0}^{M-1} (\beta - \alpha)^k \frac{f^{k+1}}{(k+1)!} + (\beta - \alpha)^M \frac{Q^{(M)}(\alpha)}{M!} \\
f(\beta) &= f(\alpha) + (\beta - \alpha) Q(\alpha) \\
&= f(\alpha) + \sum_{k=0}^{M-1} (\beta - \alpha)^{k+1} \frac{f^{k+1}}{(k+1)!} + (\beta - \alpha)^{M+1} \frac{Q^{(M)}(\alpha)}{M!} \\
&= (\beta - \alpha)^0 \frac{f(\alpha)}{0!} + \sum_{k=1}^{M} (\beta - \alpha)^k \frac{f^k}{(k)!} + (\beta - \alpha)^{M+1} \frac{Q^{(M)}(\alpha)}{M!} \\
&= \sum_{k=0}^{M} (\beta - \alpha)^k \frac{f^k}{(k)!} + (\beta - \alpha)^{M+1} \frac{Q^{(M)}(\alpha)}{M!} \\
&= P(\beta) + (\beta - \alpha)^{M+1} \frac{Q^{(M)}(\alpha)}{M!}
\end{aligned}
$$

Rudin ch 5 ex 22

(a) Suppose f is differentiable. Then $g(x) = f(x) - x$ and $g'(x) = f'(x) - 1$. If f had at least two fixed points $a < b$, then $g(a) = g(b)$. Then by rolle's theorem $g'(c) = 0 = f'(c) - 1$ for some $c \in (a, b)$ so $f'(c) = 1$. Therefore f cannot have more than 1 fixed point.

(b) Note that $f(t) - t = (1 + e^t)^{-1} > 0$, so there can be no fixed point.

(c) Then for any $a < b$, by the mean value theorem, there must be c such that

$$
\left| \frac{f(b) - f(a)}{b - a} \right| = |f'(c)| \leq A
$$

So this bound applies for all $a < b$, though the particular value may differ.

$$
\begin{aligned}
|f(b) - f(a)| &\leq A|b - a| \\
|f(x_{n+1}) - f(x_n)| &\leq A|x_{n+1} - x_n| \\
|x_{n+2} - x_{n+1}| &\leq A|x_{n+1} - x_n| \\
|x_{n+1} - x_n| &\leq A^n |x_1 - x_0|
\end{aligned}
$$

Which shows that $\lim_{n \to \infty} |x_{n+1} - x_n| = 0$ and therefore $x_n \to x$ for some $x \in \mathbb{R}$. Then using continuity of f:

$$
\begin{aligned}
\lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} x_{n+1} \\
f(x) = f(\lim_{n \to \infty} x_n) &= x
\end{aligned}
$$

So x is indeed a fixed point.

(d) It's rather difficult to draw in latex. But visually in $\mathbb{R}^2$ there would be a sequence of shrinking circles around each point in the sequence which would contain the subsequent point in the sequence. Or using the graph of the function you could draw some kind of shrinking staircase.