# Math 104 HW 2

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## 9.9

(a) For any number $M$ there exists $k$ such that $s_n > M$ for $n \geq k$ then $t_n \geq s_n > M$ for all $n \geq k$.

(b) By a symmetric argument to the above.

(c) Note that $t_n - s_n \geq 0$ and therefore. Then suppose $\lim(t_n - s_n) < 0$. But then for some $N$ there must exist $\lim(t_n - s_n) \leq t_N - s_N \leq \dfrac{\lim(t_n - s_n)}{2} < 0$, which is impossible. Therefore $\lim t_n - \lim s_n = \lim(t_n - s_n) \geq 0$ as desired.

## 9.15

Note that for $n \geq a + 1$ I have

$$\frac{a^n}{n!} \leq \frac{a^a}{a!}\left(\frac{a}{a+1}\right)^{n-a}$$

As all of the terms after $\dfrac{a}{\lceil a+1 \rceil} < 1$ will continue decreasing and the geometric series will go to 0.

## 10.7

You can construct such a sequence by taking points increasingly near the supremum. Let $\varepsilon_1 = 1$ for instance and then consider whether there is a point in $S$ that is within $\varepsilon_1$ of $\sup S$, which would be greater than $\sup S - \varepsilon_1$. If all points in $S$ were more than $\varepsilon_1$ from $\sup S$ then $\sup S - \varepsilon_1$ would be a smaller upper bound on $S$, a contradiction. Therefore let $s_1 \in S$ be a point satisfying $\sup S - \varepsilon_1 < s_1 \leq \sup S$. Then let $s_{n+1}$ be recursively defined by being a point with $\varepsilon_{n+1} = (S - s_n)/2$ which gives a monotone increasing sequence which decays at least exponentially fast to $\sup S$.

## 10.8

This is quite intuitive, as the average of an increasing seequence will also be increasing.

$$\begin{aligned}
\frac{\sum_{i=1}^n s_i}{n} &= \frac{n(\sum_{i=1}^n s_i) + \sum_{i=1}^n s_i}{n(n+1)} \\
&= \frac{\sum_{i=1}^n s_i + \frac{\sum_{i=1}^n s_i}{n}}{n+1} \\
&< \frac{\sum_{i=1}^n s_i + s_{n+1}}{n+1}
\end{aligned}$$

Where I use that

$$\sum_{i=1}^{n} s_i < \sum_{i=1}^{n} s_{n+1} = ns_{n+1}$$

## 10.9

(a)

$$\begin{aligned} s_2 &= \frac{2}{3} \\ s_3 &= \frac{3}{4}\frac{4}{9} \\ s_4 &= \frac{4}{5}\left(\frac{3}{4}\right)^2\left(\frac{2}{3}\right)^4 \end{aligned}$$

(b) This is a positive sequence of monotonically decreasing numbers as multiplying by a number $< 1$ reduces it. Therefore it has a limit.

(c) This is upper bounded by $(2/3)^n$ and therefore it decreases to 0.

## 10.10

(a)

$$\begin{aligned} s_2 &= \frac{2}{3} \\ s_3 &= \frac{5}{9} \\ s_4 &= \frac{14}{27} \end{aligned}$$

(b) The base cases were shown above.

$$\begin{aligned} s_n &> 1/2 \\ \frac{s_n + 1}{3} &> \frac{3}{2}\frac{1}{3} = \frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned} \frac{1}{2} &< s_n \\ \frac{1}{3} &< \frac{2}{3}s_n \\ \frac{1}{3} + \frac{1}{3}s_n &< s_n \end{aligned}$$

(d) Existence follows by the sequence being monotone and therefore by taking the limit of both sides of the recursion and letting

$$\begin{aligned} \lim s_n &= \frac{1}{3}\lim s_n + \frac{1}{3} \\ \lim s_n &= \frac{1}{2} \end{aligned}$$

## 10.11

(a) It exists because it's a monotone decreasing seequence.

(b) I'm guessing some weird number like $\pi/6$, as I don't think it decays fast enough to reach 0

## Squeeze Test

Let $\varepsilon > 0$. Then for some $N, M$, $|c_n - L|, |a_m - L| \leq \varepsilon$ for all $n \geq N$ and $m \geq M$. Therefore $|b_n - L| \leq \varepsilon$ for all $n \geq \max(N, M)$.