Ross 1.10

Prove $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ for all positive integers n.

Proof:

Let $P(n)$ be $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$

Base Case: $P(1)$

LHS $= (2\cdot 1+1)=3$

RHS $= 3\cdot(1)^2 = 3$

We have RHS = LHS

Inductive Hypothesis:

Assume $P(k)$ if true for some positive integer k

That is, $(2\cdot k+1)+(2\cdot k+3)+\cdots+(4\cdot k-1)=3k^2$

Inductive Step:

Note that we can rewrite the equation above to

$$(2(k+1)-1)+(2(k+1)+1)+\cdots+(4(k+1)-5)=3k^2$$

We then subtract $(2(k+1)-1)$ on both side and get:

$$(2(k+1)+1)+\cdots+(4(k+1)-5)=3k^2-\ (2(k+1)-1)$$

Then we add $(4(k+1)-3)$ and $(4(k+1)-1)$ to achieve our goal and get:

$$(2(k+1)+1)+\cdots+(4(k+1)-5)+(4(k+1)-3)+(4(k+1)-1)$$
$$=3k^2-\ (2(k+1)-1)+(4(k+1)-3)+(4(k+1)-1)$$

We can simplify$(4(k+1)-3)$ and $(4(k+1)-1)$ by combining them together with $(4(k+1)-5)$ and get:

$$(2(k+1)+1)+\cdots+(4(k+1)-1)=3k^2-6k+3=3(k^2-2k+1)$$

We can further simplify and get:

$$(2(k+1)+1)+\cdots+(4(k+1)-1)=3(k+1)^2$$

which is what $P(k+1)$ says.

So if $P(n)$ is true for some positive integer k, then it must also be true for $k+1$.

By mathematical induction, $(2n+1)+(2n+3)+\cdots+(4n-1)=3n^2$ for all positive integer n.

Ross 1.12

The binomial theorem asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= a^n + na^{n-1}b + \frac{1}{2}na^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

(a.) Verify the binomial theorem for $n = 1, 2, and\ 3$.
(b.) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.
(c.) Prove the binomial theorem using mathematical induction and part (b.)

Let $P(n)$ be $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$
$$= a^n + na^{n-1}b + \frac{1}{2}na^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

(a.) $P(1)$:
LHS: $(a+b)^1 = a+b$
RHS: $a^1 + 1 \cdot a^{1-1}b = a+b$ ⇒ LHS = RHS, so this is true for $n = 1$

$P(2)$:
LHS: $(a+b)^2 = a^2 + 2ab + b^2$
RHS: $a^2 + 2 \cdot a^{2-1}b + 1 \cdot a^{2-2}b^2 = a^2 + 2ab + b^2$ ⇒ LHS = RHS, so this is true for $n = 2$

$P(3)$:
LHS: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
RHS: $a^3 + 3 \cdot a^{3-1}b + 3 \cdot a^{3-2}b^2 + 1 \cdot a^{3-3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$
⇒ LHS = RHS, so this is true for $n = 3$

(b.)
$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\cdot(n-k)!} + \frac{n!}{(k-1)!\cdot(n-k+1)!} = n! \cdot \left(\frac{1}{(k-1)!\cdot k\cdot(n-k)!} + \frac{1}{(k-1)!(n-k)!\cdot(n-k+1)}\right) = n! \cdot \left(\frac{(n-k+1)+(k)}{k!(n-k+1)!}\right)$$
$$= n! \cdot \left(\frac{n+1}{k!\cdot((n+1)-k)}\right) = n! \cdot (n+1) \cdot \left(\frac{1}{k!\cdot((n+1)-k)}\right) = \left(\frac{(n+1)!}{k!\cdot((n+1)-k)}\right) = \binom{n+1}{k}$$

(c.) Base Case:
Shown in part (a)

Inductive Hypothesis:
Assume $P(k)$ is true for some positive integer k
This means that $(a+b)^k = \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i$

Inductive Step:
To get $(a+b)^{k+1}$, we multiply another $(a+b)$:
$$(a+b)^{k+1} = (a+b)\cdot\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i = \left(a\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i\right) + \left(b\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i\right)$$
$$\Rightarrow (a+b)^{k+1} = \left(\sum_{i=0}^{k}\binom{k}{i}a^{k-i+1}b^i\right) + \left(\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^{i+1}\right)$$

We can rewrite this as:
$$(a+b)^{k+1} = \left(\binom{k}{0}a^{k+1} + \sum_{i=1}^{k}\binom{k}{i}a^{k-i+1}b^i\right) + \left(\binom{k}{k}b^{k+1} + \sum_{i=0}^{k-1}\binom{k}{i}a^{k-i}b^{i+1}\right)$$

Set the index of the second summation start from 1 and get:

$$(a+b)^{k+1} = \left(\binom{k}{0} a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k-i+1} b^{i}\right) + \left(\binom{k}{k} b^{k+1} + \sum_{i=1}^{k} \binom{k}{i-1} a^{k-i+1} b^{i}\right)$$

Now, we get:

$$(a+b)^{k+1} = \left(\binom{k}{0} a^{k+1} + \binom{k}{k} b^{k+1} + \sum_{i=1}^{k} \binom{k}{i} \binom{k}{i-1} + a^{k-i+1} b^{i}\right)$$

Use part(b.) and we get:

$$(a+b)^{k+1} = \binom{k}{0} a^{k+1} + \binom{k}{k} b^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} + a^{k-i+1} b^{i}$$

Note that $\binom{k}{0} = \binom{k+1}{0} = 1$ and $\binom{k}{k} = \binom{k+1}{k+1} = 1$

Rewrite the formula:

$$(a+b)^{k+1} = \binom{k+1}{0} a^{k+1} + \binom{k+1}{k+1} b^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} + a^{k-i+1} b^{i}$$

$$\Rightarrow (a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} + a^{(k+1)-i} b^{i}$$

Since $(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} + a^{(k+1)-i} b^{i}$, $P(k+1)$ is true.

So if $P(n)$ is true for some positive integer k, then it must also be true for $k+1$.

By mathematical induction, the binomial theorem holds true for all positive integer n.

Ross 2.2

Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$

(1.) Note that $\sqrt[3]{2}$ is the solution to $x^3 - 2 = 0$
The only possible rational solutions by the Rational Zeros Theorem are ± 2
Since $\sqrt[3]{2}$ is a solution and is none of ± 2, it is not rational.

(2.) Note that $\sqrt[7]{5}$ is the solution to $x^7 - 5 = 0$
The only possible rational solutions by the Rational Zeros Theorem are ± 5
Since $\sqrt[7]{5}$ is a solution and is none of ± 5, it is not rational.

(3.) Note that $\sqrt[4]{13}$ is the solution to $x^4 - 13 = 0$
The only possible rational solutions by the Rational Zeros Theorem are ± 13
Since $\sqrt[4]{13}$ is a solution and is none of ± 13, it is not rational.

Ross 3.6

(a.) Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.
(b.) Use induction to prove:

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$

(a.) $|a + b + c| = |(a + b) + c|$

By the triangle inequality: $|(a + b) + c| \leq |a + b| + |c|$

By the triangle inequality: $|a + b| \leq |a| + |b|$
So $|a + b| + |c| \leq |a| + |b| + |c|$

Note that $|(a + b) + c| \leq |a + b| + |c|$
So $|(a + b) + c| \leq |a| + |b| + |c|$

Thus, $|a + b + c| \leq |a| + |b| + |c|$

(b.) Let $P(n)$ be

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$

Base Case: $P(1)$
It is true that $|a_1| \leq |a_1|$

Inductive Hypothesis: $P(k)$
Assume it is true for some positive integer k.
That is,

$$|a_1 + a_2 + \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k|$$

Inductive Step: $P(k + 1)$
By triangle inequality,

$$|(a_1 + a_2 + \cdots + a_k) + a_{k+1}| \leq |a_1 + a_2 + \cdots + a_k| + |a_{k+1}|$$

By the inductive hypothesis

$$|a_1 + a_2 + \cdots + a_k| + |a_{k+1}| \leq |a_1| + |a_2| + \cdots + |a_k| + |a_{k+1}|$$

So,

$$|(a_1 + a_2 + \cdots + a_k) + a_{k+1}| \leq |a_1| + |a_2| + \cdots + |a_k| + |a_{k+1}|$$

So if it is true for k, then it is true for $k + 1$

By the mathematical induction, $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$ for n numbers $a_1, a_2, \ldots, a_n$

Ross 4.11

Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$4.7 to show there are infinitely many rationals between a and b.

Denseness of $\mathbb{Q}$: If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

Let $a, b \in \mathbb{R}$ where $a < b$.

Let $c = \frac{a+b}{2}$

Note that since $\frac{a+b}{2} < \frac{b+b}{2}$, $c < b$
Also, $\frac{a+b}{2} > \frac{a+a}{2}$, $a < c$

Then we'll have $a < c < b$

Similarly, we can let $d = \frac{a+c}{2}$ and $e = \frac{c+b}{2}$ to get $a < d < c < e < b$

We can keep doing this infinitely times

Therefore, there are infinitely many rationals between a and b.

Ross 4.14

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a.) Prove $\sup(A + B) = supA + supB$.
(b.) Prove $\inf(A + b) = infA + infB$.

(a.) By the definition of supremum, $supA \geq a$ and $supB \geq b$
$\Longrightarrow supA + supB \geq (a + b)$

Note that since $(a + b) \in (A + B)$ and $supA + supB \geq (a + b)$ holds for arbitrary a and b, and $\sup(A + B) \in (A + B)$,

$$supA + supB \geq \sup\,(A + B)$$

Again by the definition of supremum, $\sup(A + B) \geq (a + b)$

Note that since a is any arbitrary number in A and b is any arbitrary number in B, and $supA$ is a number in A and $supB$ is a number in B, $\sup(A + B) \geq supA + supB$ holds true.

We have

$$supA + supB \geq \sup\,(A + B)$$

and

$$\sup(A + B) \geq supA$$

So, $\sup(A + B) = supA + supB$.

(b.) By the definition infimum, $infA \leq a$ and $infB \leq b$
$\Longrightarrow infA + infB \leq (a + b)$

Note that since $(a + b) \in (A + B)$ and $infA + infB \leq (a + b)$ holds for arbitrary a and b, and $\inf(A + B) \in (A + B)$,

$$infA + infB \leq \inf\,(A + B)$$

Again by the definition of infimum, $\inf(A + B) \leq (a + b)$

Note that since a is any arbitrary number in A and b is any arbitrary number in B, and $supA$ is a number in A and $supB$ is a number in B, $\inf(A + B) \leq infA + infB$ holds true.

We have

$$infA + infB \leq \inf\,(A + B)$$

and

$$\inf(A + B) \leq supA$$

So, $\inf(A + B) = infA + infB$.

Ross 7.5

Determine the following limits

(a.) $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$

(b.) $\lim\left(\sqrt{n^2+n} - n\right)$

(c.) $\lim\ (\sqrt{4n^2+n} - 2n)$

(a.) $\lim \sqrt{n^2+1} - n$

As $n \to \infty$, we can ignore the 1 in the square root and get:

$$\lim \sqrt{n^2+1} - n = \lim \sqrt{n^2} - n = \lim n - n = 0$$

(b.) $\lim\left(\sqrt{n^2+n} - n\right)$

As $n \to \infty$, n^2grows much faster than n, and hence we can ignore n and get:

$$\lim \sqrt{n^2+n} - n = \lim \sqrt{n^2} - n = \lim n - n = 0$$

(c.) $\lim\left(\sqrt{4n^2+n} - 2n\right)$

As $n \to \infty$, $4n^2$grows much faster than n, and hence we can ignore n and get:

$$\lim \sqrt{4n^2+n} - 2n = \lim \sqrt{4n^2} - 2n = \lim 2n - 2n = 0$$