Ross 1.10

We'd like to show that $(2n+1)+(2n+3)+...+(4n-1)=3n^2$ for $n \in \mathbb{Z}^+$.
Base Case: $n=1$

$$2(1)+1=3=3(1)^2$$

as desired.
Inductive Step: Now we assume that for some fixed n, $(2n+1)+(2n+3)+...+(4n-1)=3n^2$. We have:

$$(2(n+1)+1)+(2(n+1)+3)+...+(4(n+1)-1)$$
$$=(2n+3)+(2n+5)+...+(4n+1)+(4n+3)$$
$$=3n^2-(2n+1)+(4n+1_+(4n+3)$$
$$=3n^2+6n+3$$
$$=3(n+1)^2$$

where the third equality holds via our inductive hypothesis.

Ross 1.12

a.

$$(a+b)^1 = \binom{1}{0}a + \binom{1}{1}b = a + b$$

$$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + a^3$$

b.

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!}$$
$$= n!\left(\frac{1}{(n-k)!k!} + \frac{1}{(n-k+1)!(k-1)!}\right)$$
$$= n!\left(\frac{n-k+1}{(n-k+1)!k!} + \frac{k}{(n-k+1)!k!}\right) = n!\left(\frac{n+1}{(n-k+1!k!}\right)$$
$$= \binom{n+1}{k}$$

c. We start with our base case (which we've already verified in part (a). We proceed assuming the binomial theorem holds for some fixed n.

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\left(\binom{n}{0}a^n + ... + \binom{n}{n}b^n\right)$$
$$= \binom{n}{0}a^{n+1} + \left(\binom{n}{0}a^nb + \binom{n}{1}a^nb\right) + ... + \binom{n}{n}b^{n+1}$$

The inner grouped term is a result of combining like terms, which appear when multiplying one term by a and another by b yields a "like" monomial. We recognize that $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1}$, and also use the identity in part (b) to simplify our earlier expression to:

$$\binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^nb + ... + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^{n+1}$$

which is the desired result.

Ross 2.1

We want to show that each of the following numbers is irrational. We use the rational roots theorem to classify all possible rational roots for each numbers' corresponding polynomial, and after checking that none of the possible roots are valid solutions to the polynomial, we conclude that the number is not rational.

1. $x = \sqrt{3}$, so $x^2 - 3 = 0$, the only possible rational roots of this polynomial are $[\pm 3, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

2. $x = \sqrt{5}$, so $x^2 - 5 = 0$, the only possible rational roots of this polynomial are $[\pm 5, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

3. $x = \sqrt{7}$, so $x^2 - 7 = 0$, the only possible rational roots of this polynomial are $[\pm 7, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

4. $x = \sqrt{24}$, so $x^2 - 24 = 0$, the only possible rational roots of this polynomial are $[\pm 24, \pm 12, \pm 8, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

5. $x = \sqrt{31}$, so $x^2 - 31 = 0$, the only possible rational roots of this polynomial are $[\pm 31, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

Ross 2.2

We want to show that each of the following numbers is irrational. We use the rational roots theorem to classify all possible rational roots for each numbers' corresponding polynomial, and after checking that none of the possible roots are valid solutions to the polynomial, we conclude that the number is not rational.

1. $x = \sqrt[3]{2}$, so $x^3 - 2 = 0$, the only possible rational roots of this polynomial are $[\pm 2, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.
2. $x = \sqrt[7]{5}$, so $x^7 - 5 = 0$, the only possible rational roots of this polynomial are $[\pm 5, \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.
3. $x = \sqrt[4]{13}$, so $x^4 - 13 = 0$, the only possible rational roots of this polynomial are $[\pm 14, \pm 7, \pm 2 \pm 1]$, and we can verify that neither satisfy the given polynomial, thus x must be irrational.

Ross 2.7

1. We want to show that $\sqrt{4+2\sqrt{3}}-\sqrt{3}$ is rational.

$$\left(\sqrt{4+2\sqrt{3}}-\sqrt{3}\right)=x$$

$$\left(\sqrt{4+2\sqrt{3}}\right)=x+\sqrt{3}$$

$$4+2\sqrt{3}=x^2+2x\sqrt{3}+3$$

   $x=1$ satisfies this equation, so we conclude that $\sqrt{4+2\sqrt{3}}-\sqrt{3}=1$

2. We want to show that $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ is rational.

$$\left(\sqrt{6+4\sqrt{2}}-\sqrt{2}\right)=x$$

$$\left(\sqrt{6+4\sqrt{2}}\right)=x+\sqrt{2}$$

$$6+4\sqrt{2}=x^2+2x\sqrt{2}+2$$

   $x=2$ satisfies this equation, so we conclude that $\sqrt{6+4\sqrt{2}}-\sqrt{2}=2$

Ross 3.6

a.

$$|a+b+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$$

b. The base case is trivial, $|x_1| \leq |x_1|$. Assume for fixed n that

$$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$$

Then we have that

$$|x_1 + x_2 + ... + x_n + x_{n+1}| \leq |x_1 + x_2 + ... + x_n| + |x_{n+1}|$$

by the triangle inequality. Then we apply the inductive hypothesis to see that

$$|x_1 + x_2 + ... + x_n| + |x_{n+1}| \leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|$$

We combine the above inequalities to conclude that

$$|x_1 + x_2 + ... + x_n + x_{n+1}| \leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|$$

Ross 4.11

We'd like to show that for any $a, b \in \mathbb{R}$, given that $a < b$, there are infinitely many rationals r such that $a < r < b$. The denseness of $\mathbb{Q}$ gives us one such r such that $a < r < b$ (4.7 in Ross). Set $s_0 = r$. Recursively define s_{n+1} as a rational number between a and s_n (again, given by denseness of $\mathbb{Q}$). We continue this infinitely.

One can show via induction that every s_i is greater than a and less than b, but it's kind of obvious by construction. Intuitively, we're finding a rational number between a and b, and then finding a smaller rational number greater than a, and then finding an even smaller one, and so on.

Ross 4.14

a. We want to show that $\sup(A+B) = \sup A + \sup B$. We know that $\sup A + \sup B$ is an upper bound for $A+B$, because we can add the following inequalities

$$\forall x \in A, x \leq \sup A$$

$$\forall y \in B, y \leq \sup B$$

Now we have to show that $\sup A + \sup B$ is the least upper bound. Assume for contradiction that it's not the least upper bound. Then:

$$\sup A + \sup B - \sup(A+B) > 0$$

Let's call this quantity $d = \sup A + \sup B - \sup(A+B)$. We know that since $\sup A$ and $\sup B$ are least upper bounds, then $\sup A - \frac{d}{2}$ is *not* an upper bound of A, and similarly $\sup B - \frac{d}{2}$ is *not* an upper bound of B. This means there is an $a \in A$, and a $b \in B$ such that:

$$a > \sup A - \frac{d}{2}$$

$$b > \sup B - \frac{d}{2}$$

If we add these inequalities together, we get that

$$a + b > \sup A + \sup B - d$$

and after substituting d, we get that

$$a + b > \sup A + \sup B - (\sup A + \sup B - \sup(A+B) = \sup(A+B)$$

which is a contradiction, because we see that an element of $A+B$ is larger than the supremum of the set. Therefore $\sup A + \sup B$ must be the least upper bound, as desired.

b. We can use a very similar proof to show that $\inf(A+B) = \inf A + \inf B$. Again, we know that $\inf A + \inf B$ is a lower bound by adding together the inequalities from the definition of infimum. Again suppose for contradiction that the greatest lower bound of $A+B$ is larger than $\inf A + \inf B$. Set:

$$d = \inf(A+B) - \inf A - \inf B$$

We use the fact that $\inf A + \frac{d}{2}$ is not a lower bound of A, and $\inf b + \frac{d}{2}$ isn't a lower bound to B to again see that for some $a \in A, b \in B$:

$$a + b < \inf A + \inf B + d = \inf(A+B)$$

which is a contradiction because an element of $A+B$ is smaller than it's greatest lower bound. Therefore $\inf A + \inf B = \inf(A+B)$.

Ross 7.5

a.

$$\lim_{n\to\infty} \sqrt{n^2+1} - n = \lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right)\left(\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right) = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = 0$$

b.

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} n\sqrt{1+\frac{1}{n}} - n = \lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\sqrt{1+\frac{1}{n}} - 1\right)\left(\frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}+1}\right) = \lim_{n\to\infty}\left(\frac{n * \frac{1}{n}}{\sqrt{1+\frac{1}{n}}+1}\right) = \frac{1}{2}$$

c.

$$\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \lim_{n\to\infty} 2n\sqrt{1+\frac{1}{4n}} - 2n = \lim_{n\to\infty} 2n\left(\sqrt{1+\frac{1}{4n}} - 1\right)$$

$$= \lim_{n\to\infty} 2n\left(\sqrt{1+\frac{1}{4n}} - 1\right)\left(\frac{\sqrt{1+\frac{1}{4n}}+1}{\sqrt{1+\frac{1}{4n}}+1}\right) = \lim_{n\to\infty}\left(\frac{2n * \frac{1}{4n}}{\sqrt{1+\frac{1}{n}}+1}\right) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right) = \frac{1}{4}$$