Ross 9.9

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$

(a.) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$

(b.) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$

(c.) Prove that if $\lim s_n$ and $\lim t_n$ exists, then $\lim s_n \leq \lim t_n$

(a.) Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
Since $\lim s_n = +\infty$, for each $M > 0$, there is a number N that for $n > N$, $s_n > M$.
Let $N' = \max\{N, N_0\}$, then for all $n > N'$, $t_n \geq s_n > M$.
So, $\lim t_n = +\infty$.

(b.) Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
Since $\lim t_n = -\infty$, for each $M < 0$, there is a number N that for $n > N$, $t_n < M$.
Let $N' = \max\{N, N_0\}$, then for all $n > N'$, $s_n \leq t_n < M$.
So, $\lim s_n = -\infty$.

(c.) Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.
Also suppose that $\lim s_n$ and $\lim t_n$ exist.
Then, $t_n - s_n \geq 0$ for all $n > N_0$.
This implies that $\lim t_n - \lim s_n \geq 0$ for all $n > N_0$.
Which implies that $\lim t_n \geq \lim s_n$

Ross 9.15

Show $\lim_{n \to \infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

Note that as $n \to \infty$, n will eventually larger than a

In fact, after $n = a + 1$, every n will larger than a

Thus, making $\frac{a}{n}$ become smaller and smaller and eventually goes to zero.

Ross 10.7

Let S be bounded nonempty subset of $\mathbb{R}$ such that $supS$ is not in S. Prove there is a sequence (s_n) of points in S such that $lims_n = supS$

Let S be bounded nonempty subset of $\mathbb{R}$ such that $supS \notin S$.

By definition, for all $\epsilon > 0$, there exists $s \in S$ such that $s > supS - \epsilon$.

Note that since $\frac{1}{n} > 0$ for all $n > 0$, we can let $\epsilon = \frac{1}{n}$ and get $s > supS - \frac{1}{n}$

Also note that $supS > s$ for all $s \in S$ in this case.

Then, we get the relation: $supS - \frac{1}{n} < s < supS$ for all $s \in S$.

From above, we can see that (s_n) is bounded by some sequences, says $(a_n) = supS$ and $(b_n) = supS - \frac{1}{n}$.

Therefore, there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Ross 10.8

Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

Since (s_n) is an increasing sequence of positive numbers, the smallest possible sum of them is $1 + 2 + 3 + \cdots + n = \Sigma_{i=1}^{n} s_i$.

Any other possible sequence will only make $(s_1 + s_2 + \cdots + s_n)$ larger.

So, as long as $\frac{1}{n}\Sigma_{i=1}^{n} s_i$ is increasing, (σ_n) is increasing.

$\frac{\Sigma_{i=1}^{n} s_i}{n} = \frac{\left(\frac{1}{2}n(n+1)\right)}{n} = \frac{n+1}{2} \geq 1$ since $n \geq 1$.

Since the ration is greater than or equal to 1 with equality holds when $n = 1$, the sequence does increasing.

Therefore, (σ_n) is an increasing sequence.

Ross 10.9

Let $s_1 = 1$ and $s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$ for $n \geq 1$.

(a.) Find s_2, s_3 and s_4.w

(b.) Show $\lim\ s_n$ exists.

(c.) Prove $\lim s_n = 0$.

(a.) $s_2 = \left(\frac{1}{2}\right) \cdot 1^2 = \frac{1}{2}$

$s_3 = \left(\frac{2}{3}\right) \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}$

$s_4 = \left(\frac{3}{4}\right) \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}$

(b.) Note that $\left(\frac{n}{n+1}\right) \geq 0$ and therefore, $s_n \geq 0$

Furthermore, $\left(\frac{n}{n+1}\right) < 1$ since $n + 1 > n$ for all $n \geq 1$ and therefore $s_n < 1$

So we have: $0 \leq \left(\frac{n}{n+1}\right) s_n^2 < 1$ and $\left(\frac{n}{n+1}\right) s_n^2$ is decreasing.

Therefore, $\lim\ s_n$ exists.

(c.) From above we can see that both $\left(\frac{n}{n+1}\right)$ and s_n^2 is getting closer and closer to zero.

Therefore, $\lim s_n \rightarrow 0$.

Squeeze Lemma

Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq c_n$. If $L = \lim a_n = lim\ c_n$, then $\lim b_n = L$.

Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq c_n$.

Let $L = \lim a_n = \lim\ c_n$.

Then for all $\epsilon > 0$, there exists an N_a such that for all $n > N_a$, $\left| a_n - L \right| < \epsilon$.

That is, $L - \epsilon < a < L + \epsilon$.

Similarly, we can have $L - \epsilon < c < L + \epsilon$ with some N_c

Let $N_b = \max \{N_a, N_c\}$

Then we have: $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$

Rewrite above and we get: $L - \epsilon < b_n < L + \epsilon$

This implies that $\left| b_n - L \right| < \epsilon$.

Therefore, $\lim b_n = L$.