Problem 1

Ross 33.4

Solution

Give an example of a function f on [0, 1] that is not integrable for which $|f|$ is integrable.

Let f be the following: on $[a, b]$ f= $\begin{cases} 1 & x \in [0,1]/\mathbb{Q} \\ -1 & x \in [0,1] \cap \mathbb{Q} \end{cases}$

f is not integrable:

for partition: P = $\{t_0, ...t_n\}$

$U(f, P) = \sum_{i=0}^{n}$

$sup(f(x) : x \in [t_i, t_{i+1}]) * (t_{i+1} - t_i) = 1$

$L(f, P) = \sum_{i=0}^{n} \inf(f(x) : x \in [t_i, t_{i+1}]) * (t_{i+1} - t_i) = 0$

Observe that $|f| = 1 \forall x \in [a, b]$ which is integrable

$\sup(1 : x \in [t_i, t_{i+1}]) = 1 = \inf(1 : x \in [t_i, t_{i+1}])$

$|\sup(f(x) : x \in [t_i, t_{i+1}])| = \inf(f(x) : x \in [t_i, t_{i+1}]) \forall x \in [0, 1]$

Problem 2

Ross 33.7

Solution

Let f be a bounded function on [a, b], so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

a) Show $U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$ for all partitions P of [a, b]. Hint: $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.

$U(f^2, P) = \sum_{i=0}^{n} \sup(f(x)^2 : x \in [t_i, t_{i+1}]) * (t_{i+1} - t_i)$

$L(f^2, P) = \sum_{i=0}^{n} \inf(f(x)^2 : x \in [t_i, t_{i+1}]) * (t_{i+1} - t_i)$

$U(f^2, P) - L(f^2, P) = \sum_{i=0}^{n} [\sup(f(x)^2 : x \in [t_i, t_{i+1}]) - \inf(f(x)^2 : x \in [t_i, t_{i+1}])] * (t_{i+1} - t_i)$

We want to thus show that $[\sup(f(x)^2 : x \in [t_i, t_{i+1}]) - \inf(f(x)^2 : x \in [t_i, t_{i+1}])] \leq 2B(\sup(f(x) : x \in [t_i, t_{i+1}]) - \inf(f(x) : x \in [t_i, t_{i+1}]))$

from hint:

$|f(x)^2 - f(y)^2| = [f(x) + f(y)] \cdot [f(x) - f(y)]$

on any interval $[a, b]$,

$|f(x)^2 - f(y)^2| = |[f(x) + f(y)] \cdot [f(x) - f(y)]| \leq |[f(x) + f(y)]| \cdot |[f(x) - f(y)]| \leq 2B \cdot |[f(x) - f(y)]|$

$U(f^2, P) - L(f^2, P) = \sum_{i=0}^{n} [\sup(f(x)^2 : x \in [t_i, t_{i+1}]) - \inf(f(x)^2 : x \in [t_i, t_{i+1}])] * (t_{i+1} - t_i)$

$\leq \sum_{i=0}^{n} 2B[\sup(f(x) : x \in [t_i, t_{i+1}]) - \inf(f(x) : x \in [t_i, t_{i+1}])] * (t_{i+1} - t_i) = 2B(U(f, P) - L(f, P))$

b) Show that if f is integrable on [a, b], then f^2 also is integrable on [a, b].

Want to show: $U(f, P) - L(f, P) \leq \epsilon_0$ implies

$U(f^2, P) - L(f^2, P) \leq \epsilon$

let $\epsilon_0 = \frac{\epsilon}{2B}$

We have already show that:

$U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) \leq \epsilon$

Problem 3

Ross 33.13

Solution $\int_a^b f = \int_a^b g$

by intermediate value theorem, we can show that there exists a $x \in [a, b]$ s.t. $f(x) = g(x)$

Since $\int_a^b f - g = 0$, there exists a $x \in [a, b]$ s.t.

$fg = \frac{1}{b-a} \int_a^b f - g = 0$

Problem 4

Ross 35.4

Solution $F(t) = sin(t)$ for $t \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

a) $\int_0^{\frac{\pi}{2}} xdF(x)$

by integration by parts(since both x and $\sin(x)$ are increasing on $[0, \frac{\pi}{2}]$: $\int_a^b F_1 dF_2 = F_1(b)F_2(b) - F_1(a)F_2(a) - \int_a^b F_2 dF_1$.

$\int_0^{\frac{\pi}{2}} xdF(x) = \frac{\pi}{2} * sin(\frac{\pi}{2}) - 0 - \int_0^{\frac{\pi}{2}} sin(x)dx = \frac{\pi}{2} - 0 - [0-1] = \frac{\pi}{2} + 1$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xdF(x) = \frac{\pi}{2} * sin(\frac{\pi}{2}) + \frac{\pi}{2} sin(-\frac{\pi}{2}) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin(x)dx = \frac{\pi}{2} + \frac{\pi}{2} - [0-0] = 1$

Problem 4

Ross 35.9(a)

Solution Show $\int_a^b f dF = f(x)[F(b) - F(a)]$ for some $x \in [a, b]$

Inspiration by peers:

We can bound the intergral by $U(f, F, P)$ and $L(f, F, p)$

$U(f, F, P) = \sum_{i=1}^{n} sup(f, [t_i, t_{i-1}])(F(t_i) - F(t_{i-1})) \leq \sup(f, [a, b])(F(b) - F(a))$

$L(f, F, P) = \sum_{i=1}^{n} inf(f, [t_i, t_{i-1}])(F(t_i) - F(t_{i-1})) \geq \inf(f, [a, b])(F(b) - F(a))$

the integral itself lies between the L and U

$\inf(f, [a, b])(F(b) - F(a)) \leq \int_a^b f dF \leq \sup(f, [a, b])(F(b) - F(a))$

Divide by $(F(b) - F(a))$, and wlog assume $y > x$

$f(x) = \inf(f, [a, b]) \leq \frac{1}{(F(b)-F(a))} \int_a^b f dF \leq \sup(f, [a, b]) = f(y)$

By the Intermediate Value Theorem, since f is continuous there must exist a point c between [x,y]

$f(c) = \frac{1}{(F(b)-F(a))} \int_a^b f dF$