

Problem 1.10

Prove $(2n + 1) + (2n + 3) + (2n + 5) + \dots + (4n - 1) = 3n^2$ for all positive integers n .

Solution

Rewrite: $(2n + 1) + (2n + 3) + (2n + 5) + \dots + (4n - 1) = \sum_{i=1}^n (2n - 1 + 2i)$

Base case: $n = 1$

$$2(1) + 1 = 3(1)^2$$

Inductive hypothesis

Start by splitting the sum into terms $1-n$ and $n+1$

$$\begin{aligned} \sum_{i=1}^{n+1} (2(n+1) - 1 + 2i) &= \sum_{i=1}^{n+1} (2n - 1 + 2i + 2) = 2(n+1) + \sum_{i=1}^{n+1} (2n - 1 + 2i) \\ &= 2(n+1) + (2n - 1 + 2(n+1)) + \sum_{i=1}^n (2n - 1 + 2i) \end{aligned}$$

Substitute what we have proved for the n case:

$$= 2(n+1) + (2n - 1 + 2(n+1)) + 3n^2$$

Simplify

$$= 3n^2 + 6n + 3 = 3(n^2 + 2n + 1) = 3(n+1)^2$$

Problem 1.12

- a) Verify the binomial theorem for $n = 1, 2$, and 3 .
- b) Show for $k = 1, 2, \dots, n$. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$
- c) Prove the binomial theorem using mathematical induction and part (b).

Solution

a) $n = 1$

$$(a + b) = \binom{1}{0}a + \binom{1}{1}b = a + b$$

$n = 2$

$$(a + b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$$

$n = 3$

$$(a + b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

b) Expand:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

shift factorials

$$= \frac{n!}{k*(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)*(n-k)!}$$

multiply by 1 to clean up denominators

$$= \frac{n!(n-k+1)}{k*(k-1)!(n-k+1)*(n-k)!} + \frac{n!*k}{k*(k-1)!(n-k+1)*(n-k)!}$$

$$= \frac{n!(n-k+1+k)}{k*(k-1)!(n-k+1)*(n-k)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$$

c) Prove the binomial theorem using mathematical induction and part (b).

Base case:

in part a), we proved for some cases of n .

Inductive hypothesis:

$$(a + b)^n = a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n$$

Inductive step:

$$\text{We want to show } (a + b)^{n+1} = a^{n+1} + \sum_{i=1}^n \binom{n+1}{i} a^{n+1-i} b^i + b^{n+1}$$

$$(a + b)^{n+1} = (a + b) * (a + b)^n$$

Distributing:

$$= a(a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n) + b(a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n)$$

$$\begin{aligned}
&= a^{n+1} + \sum_{i=1}^{n-1} \binom{n}{i} a^{n+1-i} * b^i + ab^n + ba^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} + b^{n+1} \\
&= a^{n+1} + \sum_{i=1}^n \binom{n}{i} a^{n+1-i} * b^i + \sum_{i=0}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} + b^{n+1}
\end{aligned}$$

Recognise that we can alter the index slightly to get the following:

$$\sum_{i=0}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} = \sum_{i=1}^n \binom{n}{i-1} a^{n+1-i} * b^i$$

Substituting:

$$= a^{n+1} + \sum_{i=1}^n \binom{n}{i} a^{n+1-i} * b^i + \sum_{i=1}^n \binom{n}{i-1} a^{n+1-i} * b^i + b^{n+1}$$

combining under one sum:

$$= a^{n+1} + \sum_{i=1}^n \binom{n}{i} a^{n+1-i} * b^i + \binom{n}{i-1} a^{n+1-i} * b^i + b^{n+1}$$

factoring:

$$= a^{n+1} + \sum_{i=1}^n (\binom{n}{i} + \binom{n}{i-1}) a^{n+1-i} b^i + b^{n+1}$$

Thus, we have shown the inductive step holds true.

$$= a^{n+1} + \sum_{i=1}^n \binom{n+1}{i} a^{n+1-i} b^i + b^{n+1} = (a + b)^{n+1}$$

Problem 2.1

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ are irrational

Solution

1. $\sqrt{3}$

$$x^2 - 3 = 0$$

$$c_2 = 1, c_0 = 3$$

The only possible rational roots are $\pm 1, \pm 3$, neither of these solve the equation.

2. $\sqrt{5}$

$$x^2 - 5 = 0$$

$$c_2 = 1, c_0 = 5$$

The only possible rational roots are $\pm 1, \pm 5$, neither of these solve the equation.

3. $\sqrt{7}$

$$x^2 - 7 = 0$$

$$c_2 = 1, c_0 = 7$$

The only possible rational roots are $\pm 1, \pm 7$, neither of these solve the equation.

4. $\sqrt{24}$

$$x^2 - 24 = 0$$

$$c_2 = 1, c_0 = 24$$

The only possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$, none of these solve the equation.

5. $\sqrt{31}$

$$x^2 - 31 = 0$$

$$c_2 = 1, c_0 = 31$$

The only possible rational roots are $\pm 1, \pm 31$, neither of these solve the equation.

Problem 2.2

Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are irrational

Solution

1. $\sqrt[3]{2}$

$$x^3 - 2 = 0$$

$$c_2 = 1, c_0 = 2$$

The only possible rational roots are ± 1 , ± 2 , neither of these solve the equation.

2. $\sqrt[7]{5}$

$$x^7 - 5 = 0$$

$$c_2 = 1, c_0 = 5$$

The only possible rational roots are ± 1 , ± 5 , neither of these solve the equation.

3. $\sqrt[4]{13}$

$$x^4 - 13 = 0$$

$$c_2 = 1, c_0 = 13$$

The only possible rational roots are ± 1 , ± 13 , neither of these solve the equation.

Problem 2.7

Show $\sqrt{4 + 2\sqrt{3}} - \sqrt{3}$, $\sqrt{6 + 4\sqrt{2}} - \sqrt{2}$ are rational

Solution

1. $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = x$

$$(x + \sqrt{3})^2 = 4 + 2\sqrt{3}$$

$$= x^2 + 2\sqrt{3}x + 3 = 4 + 2\sqrt{3}$$

$x = 1$ satisfies the above equality, and 1 is rational and thus $\sqrt{4 + 2\sqrt{3}} - \sqrt{3} = 1$ is rational

2. $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = x$

$$(x + \sqrt{2})^2 = 6 + 4\sqrt{2}$$

$$x^2 + 2\sqrt{2}x + 2 = 6 + 4\sqrt{2}$$

$x = 2$ satisfies the above equality, and 2 is rational and thus $\sqrt{6 + 4\sqrt{2}} - \sqrt{2} = 2$ is rational.

Problem 3.6

Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$

and

$$|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$$

Solution

a) The triangle inequality states:

$$|a + b| \leq |a| + |b| \text{ for all } a, b$$

letting $b' = b + c$, which is still a real number.

$$|a + b + c| = |a + b'| \leq |a| + |b'| = |a| + |b + c|$$

We can apply the triangle inequality again to the right since b and c are still real numbers.

$$|b + c| \leq |b| + |c|$$

Thus,

$$|a + b + c| \leq |a| + |b + c| \leq |a| + |b| + |c|$$

b) the base case ($n = 3$) is prove in part a). We simple let $a_1 = a, a_2 = b, a_3 = c$

$$\text{Inductive hypothesis: } |a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$$

Inductive step:

$$|a_1 + a_2 + \dots a_n + a_{n+1}| \leq |a_1| + |a_2| + \dots |a_n| + |a_{n+1}|$$

We can combine $a_n + a_{n+1} = a'$

since a' is still a real number, we the following still holds true.

$$|a_1 + a_2 + \dots a'| \leq |a_1| + |a_2| + \dots |a'|$$

and from the triangle inequality,

$$|a'| = |a_n + a_{n+1}| \leq |a_n| + |a_{n+1}|$$

Thus, the following must also be true:

$$|a_1| + |a_2| + \dots |a'| \leq |a_1| + |a_2| + \dots |a_n| + |a_{n+1}|$$

Finally, by the ordering of the inequalities.

$$|a_1 + a_2 + \dots a_n + a_{n+1}| \leq |a_1| + |a_2| + \dots |a_n| + |a_{n+1}|$$

Problem 4.11

$a < b \in \mathbb{R}$, show there are infinitely many rationals between a and b .

Solution Denseness: If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

We can proceed with induction.

Base case ($n = 1$): by the denseness property, we know that at least 1 rational number exists between a and b .

Inductive hypothesis: there are n distinct rational numbers (denoted by $r_1 \dots r_n$) between a and b . $r_n < r_{n-1} < \dots < r_1$.

Inductive step: We know that r_n is also a real number since \mathbb{Q} is a subset of \mathbb{R} .

We can now use the denseness property of \mathbb{R} to prove the existence of another rational number

Denseness: $a < r_{n+1} < r_n$

by ordering of the inequalities, $a < r_{n+1} < b$

We can continue for as long as we have natural numbers n , so there are an infinite number of rational numbers between a and b .

Problem 4.14

Prove $\sup(A+B) = \sup(A) + \sup(B)$

and

$\inf(A + B) = \inf A + \inf B$.

Solution

a) By construction $\sup(A + B)$ contains all the elements $a + b$.

We can construct an upper bound for A:

$$\sup(A + B) \geq a + b$$

$$\sup(A + B) - b \geq a, \forall a, b \in A, B$$

This is true because $A + B$ contains all the sums $a + b$ and uses definition of sup,

$$\text{Thus, } \sup(A) \leq \sup(A + B) - b, \forall b \in B$$

Now, we can use the hint to show

$$\sup(A + B) - \sup(A) \geq \sup(A + B) - \sup(A + B) + b = b, \forall b \in B$$

$$\sup(A + B) - \sup(A) \geq b, \forall b \in B$$

$$\sup(A + B) \geq \sup(A) + b, \forall b \in B$$

Together, since $\sup(B) \geq b, \forall b \in B$

$$\sup(A + B) = \sup(A) + \sup(B)$$

b)

We can construct a lower bound for A:

$$\inf(A + B) - b \leq a, \forall a, b \in A, B$$

$$\inf(A + B) \leq a + b, \forall a, b \in A, B$$

This equality is true because $A + B$ contains all the sums $a + b$ and uses def of inf,

$$\text{Thus, } \inf(A + B) - b \leq \inf(A), \forall b \in B$$

Now, like above we can use the above relationship to show the following relationship between $\inf(A)$ and $\inf(A+B)$

$$\inf(A + B) - \inf(A) \leq \inf(A + B) - \inf(A + B) - b = b, \forall b \in B$$

$$\inf(A + B) - \inf(A) \leq b, \forall b \in B$$

$$\inf(A + B) \leq \inf(A) + b, \forall b \in B$$

Together, since $\inf(B) \leq b, \forall b \in B$

$$\inf(A + B) = \inf(A) + \inf(B)$$

Problem 7.5

limits

Solution

1. $\lim \sqrt{n^2 + 1} - n$

We will 'rationalize the denominator'

$$\begin{aligned} (\sqrt{n^2 + 1} - n) \frac{\sqrt{n^2 + 1} + n}{\sqrt{n^2 + 1} + n} &= \frac{n^2 + 1 - n^2}{\sqrt{n^2 + 1} + n} \\ &= \frac{1}{\sqrt{n^2 + 1} + n} \end{aligned}$$

Since there is a constant term on top and a term that increases as a function of n on the bottom,

$$\lim \frac{1}{\sqrt{n^2 + 1} + n} = 0$$

2. $\lim \sqrt{n^2 + n} - n$

$$\begin{aligned} (\sqrt{n^2 + n} - n) \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} &= \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} \\ &= \frac{n}{\sqrt{n^2 + n} + n} \\ &= \frac{1}{\frac{\sqrt{n^2 + n}}{n} + 1} \end{aligned}$$

Taking the limit of the only term with n,

$$\lim \frac{\sqrt{n^2 + n}}{n} = \lim \frac{n\sqrt{1 + \frac{1}{n}}}{n} = \lim \sqrt{1 + \frac{1}{n}} = 1$$

Thus, after substituting back in...

$$\lim \sqrt{n^2 + n} - n = \frac{1}{2}$$

3. $\lim \sqrt{4n^2 + n} - 2n$

$$\begin{aligned} (\sqrt{4n^2 + n} - 2n) \frac{\sqrt{4n^2 + n} + 2n}{\sqrt{4n^2 + n} + 2n} &= \frac{4n^2 + n - 4n^2}{\sqrt{4n^2 + n} + 2n} = \frac{n}{\sqrt{4n^2 + n} + 2n} \\ &= \frac{1}{\frac{\sqrt{4n^2 + n}}{n} + 2} \end{aligned}$$

Taking the limit of the only term with n,

$$\lim \frac{\sqrt{4n^2 + n}}{n} = \lim \frac{2n\sqrt{1 + \frac{1}{4n}}}{n} = 2 * \lim \sqrt{1 + \frac{1}{4n}} = 2 * 1$$

Thus, after substituting back in...

$$\lim \sqrt{4n^2 + n} - 2n = \frac{1}{4}$$