Problem 1.10

Prove $(2n+1)+(2n+3)+(2n+5)+...+(4n-1)=3n^2$ for all positive integers n.

Solution

Rewrite: $(2n+1)+(2n+3)+(2n+5)+...+(4n-1)=\sum_{i=1}^{n}(2n-1+2i)$

Base case: n = 1

$2(1)+1=3(1)^2$

Inductive hypothesis

Start by splitting the sum into terms 1-n and n+1

$\sum_{i=1}^{n+1}(2(n+1)-1+2i)=\sum_{i=1}^{n+1}(2n-1+2i+2)=2(n+1)+\sum_{i=1}^{n+1}(2n-1+2i)$

$=2(n+1)+(2n-1+2(n+1))+\sum_{i=1}^{n}(2n-1+2i)$

Substitute what we have proved for the n case:

$=2(n+1)+(2n-1+2(n+1))+3n^2$

Simplify

$=3n^2+6n+3=3(n^2+2n+1)=3(n+1)^2$

Problem 1.12

a) Verify the binomial theorem for n = 1, 2, and 3.

b) Show for k = 1, 2, ... , n. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

c) Prove the binomial theorem using mathematical induction and part (b).

Solution

a) n = 1

$(a+b) = \binom{1}{0}a + \binom{1}{1}b = a + b$

n = 2

$(a+b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$

n = 3

$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = a^3 + 3a^2b + 3ab^2 + b^2$

b) Expand:

$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$

shift factorials

$= \frac{n!}{k*(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)*(n-k)!}$

multiply by 1 to clean up denominators

$= \frac{n!(n-k+1)}{k*(k-1)!(n-k+1)*(n-k)!} + \frac{n!*k}{k*(k-1)!(n-k+1)*(n-k)!}$

$= \frac{n!(n-k+1+k)}{k*(k-1)!(n-k+1)*(n-k)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$

c) Prove the binomial theorem using mathematical induction and part (b).

Base case:

in part a), we proved for some cases of n.

Inductive hypothesis:

$(a+b)^n = a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n$

Inductive step:

We want to show $(a+b)^{n+1} = a^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} a^{n+1-i} b^i + b^{n+1}$

$(a+b)^{n+1} = (a+b) * (a+b)^n$

Distributing:

$= a(a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n) + b(a^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^i + b^n)$

$= a^{n+1} + \sum_{i=1}^{n-1} \binom{n}{i} a^{n+1-i} * b^i + ab^n + ba^n + \sum_{i=1}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} + b^{n+1}$

$= a^{n+1} + \sum_{i=1}^{n} \binom{n}{i} a^{n+1-i} * b^i + \sum_{i=0}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} + b^{n+1}$

Recognise that we can alter the index slightly to get the following:

$\sum_{i=0}^{n-1} \binom{n}{i} a^{n-i} * b^{i+1} = \sum_{i=1}^{n} \binom{n}{i-1} a^{n+1-i} * b^i$

Substituting:

$= a^{n+1} + \sum_{i=1}^{n} \binom{n}{i} a^{n+1-i} * b^i + \sum_{i=1}^{n} \binom{n}{i-1} a^{n+1-i} * b^i + b^{n+1}$

combining under one sum:

$= a^{n+1} + \sum_{i=1}^{n} \binom{n}{i} a^{n+1-i} * b^i + \binom{n}{i-1} a^{n+1-i} * b^i + b^{n+1}$

factoring:

$= a^{n+1} + \sum_{i=1}^{n} (\binom{n}{i} + \binom{n}{i-1}) a^{n+1-i} b^i + b^{n+1}$

Thus, we have shown the inductive step holds true.

$= a^{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} a^{n+1-i} b^i + b^{n+1} = (a+b)^{n+1}$

Problem 2.1

Show $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{24}, \sqrt{31}$ are irrational

Solution

1. $\sqrt{3}$

   $x^2 - 3 = 0$

   $c_2 = 1, c_0 = 3$

   The only possible rational roots are ± 1, ± 3, neither of these solve the equation.

2. $\sqrt{5}$

   $x^2 - 3 = 0$

   $c_2 = 1, c_0 = 5$

   The only possible rational roots are ± 1, ± 5, neither of these solve the equation.

3. $\sqrt{7}$

   $x^2 - 3 = 0$

   $c_2 = 1, c_0 = 7$

   The only possible rational roots are ± 1, ± 7, neither of these solve the equation.

4. $\sqrt{24}$

   $x^2 - 3 = 0$

   $c_2 = 1, c_0 = 24$

   The only possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$, none of these solve the equation.

5. $\sqrt{31}$

   $x^2 - 3 = 0$

   $c_2 = 1, c_0 = 31$

   The only possible rational roots are ± 1, ± 31, neither of these solve the equation.

Problem 2.2

Show $\sqrt[3]{2}$, $\sqrt[7]{5}$, $\sqrt[4]{13}$ are irrational

Solution

1. $\sqrt[3]{2}$

   $x^3 - 2 = 0$

   $c_2 = 1, c_0 = 2$

   The only possible rational roots are ± 1, ± 2, neither of these solve the equation.

2. $\sqrt[7]{5}$

   $x^7 - 5 = 0$

   $c_2 = 1, c_0 = 5$

   The only possible rational roots are ± 1, ± 5, neither of these solve the equation.

3. $\sqrt[4]{13}$

   $x^4 - 13 = 0$

   $c_2 = 1, c_0 = 13$

   The only possible rational roots are ± 1, ± 13, neither of these solve the equation.

Problem 2.7

Show $\sqrt{4+2\sqrt{3}}-\sqrt{3}$, $\sqrt{6+4\sqrt{2}}-\sqrt{2}$ are rational

Solution

1. $\sqrt{4+2\sqrt{3}}-\sqrt{3}=x$

   $(x+\sqrt{3})^2=4+2\sqrt{3}$

   $=x^2+2\sqrt{3}x+3=4+2\sqrt{3}$

   x = 1 satisfies the above equality, and 1 is rational and thus $\sqrt{4+2\sqrt{3}}-\sqrt{3}=1$ is rational

2. $\sqrt{6+4\sqrt{2}}-\sqrt{2}=x$

   $(x+\sqrt{2})^2=6+4\sqrt{2}$

   $x^2+2\sqrt{2}x+2=6+4\sqrt{2}$

   x = 2 satisfies the above equality, and 2 is rational and thus $\sqrt{6+4\sqrt{2}}-\sqrt{2}=2$ is rational.

Problem 3.6

Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in R$

and

$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$

Solution

a) The triangle inequality states:

$|a+b| \leq |a|+|b|$ for all a, b

letting $b' = b+c$, which is still a real number.

$|a+b+c| = |a+b'| \leq |a|+|b'| = |a|+|b+c|$

We can apply the triangle inequality again to the right since b and c are still real numbers.

$|b+c| \leq |b|+|c|$

Thus,

$|a+b+c| \leq |a|+|b+c| \leq |a|+|b|+|c|$

b) the base case (n = 3) is prove in part a). We simple let $a_1 = a, a_2 = b, a_3 = c$

Inductive hypothesis: $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$

Inductive step:

$|a_1 + a_2 + ...a_n + a_{n+1}| \leq |a_1| + |a_2| + ...|a_n| + |a_{n+1}|$

We can combine $a_n + a_{n+1} = a'$

since a' is still a real number, we the following still holds true.

$|a_1 + a_2 + ...a'| \leq |a_1| + |a_2| + ...|a'|$

and from the triangle inequality,

$|a'| = |a_n + a_{n+1}| \leq |a_n| + |a_{n+1}|$

Thus, the following must also be true:

$|a_1| + |a_2| + ...|a'| \leq |a_1| + |a_2| + ...|a_n| + |a_{n+1}|$

Finally, by the ordering of the inequalities.

$|a_1 + a_2 + ...a_n + a_{n+1}| \leq |a_1| + |a_2| + ...|a_n| + |a_{n+1}|$

Problem 4.11

$a < b \in R$, show there are infinitely many rationals between a and b.

Solution Denseness: If $a, b \in R$ and a ¡ b, then there is a rational $r \in Q$ such that $a < r < b$.

We can proceed with induction.

Base case(n = 1): by the denseness property, we know that at least 1 rational number exists between a and b.

Inductive hypothesis: there are n distinct rational numbers (denoted by $r_1 ... r_n$) between a and b. $r_n < r_1 ... r_{n-1}$.

Inductive step: We know that r_n is also a real number since Q is a subset of R.

We can now use the denseness property of R to prove the existence of another rational number

Denseness: $a < r_{n+1} < r_n$

by ordering of the inequalities, $a < r_{n+1} < b$

We can continue for as long as we have natural numbers n, so there are an infinite number of rational numbers between a and b.

Problem 4.14

Prove sup(A+B) = sup(A) + sup(B)

and

inf(A + B) = inf A + inf B.

Solution

a) By construction $sup(A+B)$ contains all the elements a + b.

We can construct an upper bound for A:

$sup(A+B) \geq a+b$

$sup(A+B) - b \geq a, \forall a, b \in A, B$

This is true because $A+B$ contains all the sums $a+b$ and uses definition of sup,

Thus, $sup(A) \leq sup(A+B) - b, \forall b \in B$

Now, we can use the hint to show

$sup(A+B) - sup(A) \geq sup(A+B) - sup(A+B) + b = b, \forall b \in B$

$sup(A+B) - sup(A) \geq b, \forall b \in B$

$sup(A+B) \geq sup(A) + b, \forall b \in B$

Together, since $sup(A+B) - sup(A) \geq b \forall b \in B$

We can establish a supremum:

$sup(A+B) - sup(A) = sup(B)$

And thus, $sup(A+B) = sup(A) sup(B)$

b)

We can construct a lower bound for A:

$inf(A+B) - b \leq a, \forall a, b \in A, B$

$inf(A+B) \leq a+b, \forall a, b \in A, B$

This equality is true because $A+B$ contains all the sums $a+b$ and uses def of inf,

Thus, $inf(A+B) - b \leq inf(A), \forall b \in B$

Now, like above we can use the above relationship to show the following relationship between inf(A) and inf(A+B)

$inf(A+B) - inf(A) \leq inf(A+B) - inf(A+B) - b = b, \forall b \in B$

$inf(A+B) - inf(A) \leq b, \forall b \in B$

$inf(A+B) \leq inf(A) + b, \forall b \in B$

Together, since $inf(A+B) - inf(A) \leq b, \forall b \in B$

We can establish an infimum:

$inf(A+B) - inf(A) = inf(B)$

And thus, $inf(A+B) = inf(A) + inf(B)$

Problem 7.5

limits

Solution

1. $lim\sqrt{n^2+1} - n$

   We will 'irrationalize the denominator'

   $(\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$

   $= \frac{1}{\sqrt{n^2+1}+n}$

   Since there is a constant term on top and a term that increases as a function of n on the bottom,

   $lim\frac{1}{\sqrt{n^2+1}+n} = 0$

2. $lim\sqrt{n^2+n} - n$

   $(\sqrt{n^2+n} - n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$

   $= \frac{n}{\sqrt{n^2+n}+n}$

   $= \frac{1}{\frac{\sqrt{n^2+n}}{n}+1}$

   Taking the limit of the only term with n,

   $lim\frac{\sqrt{n^2+n}}{n} = lim\frac{n\sqrt{1+\frac{1}{n}}}{n} = lim\sqrt{1+\frac{1}{n}} = 1$

   Thus, after substituting back in...

   $lim\sqrt{n^2+n} - n = \frac{1}{2}$

3. $lim\sqrt{4n^2+n} - 2n$

   $(\sqrt{4n^2+n} - 2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} = \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \frac{n}{\sqrt{4n^2+n}+2n}$

   $= \frac{1}{\frac{\sqrt{4n^2+n}}{n}+2}$

   Taking the limit of the only term with n,

   $lim\frac{\sqrt{4n^2+n}}{n} = lim\frac{2n\sqrt{1+\frac{1}{4n}}}{n} = 2*lim\sqrt{1+\frac{1}{4n}} = 2*1$

   Thus, after substituting back in...

   $lim\sqrt{4n^2+n} - 2n = \frac{1}{4}$