Problem 9.9

Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$. (a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$. (b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$. (c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Solution

a) By Definition of limits,

$\forall M > 0, \exists n > N s.t. s_n > M$

since $\lim s_n = +\infty$

$s_n > M$

Since $s_n \leq t_n$

$t_n > s_n > M$

Thus, for the same n and M we have shown that

$\lim t_n = \infty$

b) By Definition of limits,

$\forall M < 0, \exists n > N s.t. t_n < M$

since $\lim t_n = -\infty$

$t_n < M$

Since $s_n \leq t_n$

$s_n \leq t_n < M$

Thus, for the same n and M we have shown that

$\lim s_n = -\infty$

c) let $\lim s_n = s$ and $\lim t_n = t$

We are given

$t_n - s_n \geq 0 \forall n > N_0$

since $t_n - s_n$ is a strictly non-negative number for all n,

$\lim(t_n - s_n) \geq 0$

we can take the limit of the difference as normal

$\lim t_n - \lim s_n \geq 0$

which implies what we want

$\lim s_n \leq \lim t_n$

Problem 9.15

Show $\lim_{x \to +\infty} \frac{a^n}{n!} = 0$ for all $a \in R$.

Solution Let L = $\lim \frac{s_{n+1}}{s_n}$

if L ¡ 1, then $\lim s_n = 0$

L = $\lim \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \lim \frac{a}{n+1} = 0$

$L < 1$

Thus, $\lim s_n = \lim \frac{a^n}{n!} = 0$

Problem 10.7

Let S be a bounded nonempty subset of R such that sup S is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Solution Let sup(s) = s

$\forall n > N, s - \frac{1}{n}$ is not necessarily an upper bound for s

define a sequence in S called s_n

and let $s - \frac{1}{n} < s_n < s$

$\lim s - \frac{1}{n} = s$

by the squeeze lemma, $\lim s_n = s$

Problem 10.8

Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + ... + s_n)$. Prove (σ_n) is an increasing sequence.

Solution We want to show that the sequence σ_n is increasing.

$\sigma_{n+1} \geq \sigma_n$

base case: n = 1

$\frac{1}{2}(s_1 + s_2) \geq \frac{1}{1}(s_1) = \frac{1}{2}(s_1 + s_1)$

since $s_2 > s_1$, the base case holds for an n

Inductive step

$\frac{1}{n+1}(s_1 + s_2 + ... + s_n + s_{n+1}) \geq \frac{1}{n}(s_1 + s_2 + ... + s_n)$

Identity:

$\frac{s_n}{n} - \frac{s_n}{n+1} = \frac{s_n}{n(n+1)}$

Applying the identity and subtracting $\frac{s_1}{n+1} + \frac{s_2}{n+1} ... \frac{s_n}{n+1}$ from both sides,

$\frac{s_{n+1}}{n+1} \geq \frac{s_1}{n(n+1)} + \frac{s_2}{n(n+1)} + ... \frac{s_n}{n(n+1)}$

$s_{n+1} \geq \frac{1}{n}(s_n + s_2 + ... s_n)$

It is true by definition of s_n being an increasing sequence that the above holds true, since $s_{n+1} > s_n > s_{n-1} ...$

Thus, σ_n is an increasing sequence.

Problem 10.9

$s_n = 1, s_{n+1} = \frac{n}{n+1} s_n^2$

Solution

a) $s_2 = \frac{2}{3} 1^2 = \frac{2}{3}$

$s_3 = \frac{3}{4} (\frac{2}{3})^2 = \frac{12}{36} = \frac{1}{3}$

$s_4 = \frac{4}{5} \frac{1}{3}^2 = \frac{4}{45}$

b) We will try induction to prove that it is a decreasing sequence.

Base case: n = 1 is proven above since

$s_2 < s_1$

$\frac{2}{3} < 1$

Inductive step:

$\frac{n+1}{n+2} < 1$

$s_{n+2} = \frac{n+1}{n+2} s_{n+1} < s_{n+1}$

Thus, we have proven that

$s_n < s_{n-1} ... < s_1$

This is a strictly decreasing sequence

We also need to show that it is bounded.

We can still use a short induction combined with our above proof.

Since s_n is decreasing, then $s_n \leq 1$

we also know that $s_n > 0 \forall n$

This is because $\frac{n}{n+1}$ will always be a positive fraction $0 < \frac{n}{n+1} < 1$

Let $s > 0, 0 < r < 1$

$s * r > 0$ holds true trivially since $s > 0$

$s * r * r$ still holds true via the same reasoning.

Thus, let $s_n = s, r = \frac{n}{n+1}$

$s_n > 0$

and thus, s_n is bounded by its first element and 0

$0 < s_{n+1} < s_n \leq 1 \forall n > 0$

c) $s = \lim_{n\to\infty} s_{n+1}$

$\lim s_{n+1} = \lim s_n$

$s_{n+1} = \frac{n}{n+1} s_n$

$s = \frac{n}{n+1} s$

s = 0 satisfies the above equation

thus:

$\lim s_n = 0$

Problem 10.10

$s_n = 1, s_{n+1} = \frac{1}{3}(s_n + 1)$

Solution

a) $s_2 = \frac{1}{3} \cdot (1+1) = \frac{2}{3}$

$s_3 = \frac{1}{3} \cdot (\frac{2}{3} + 1) = \frac{2}{9} + \frac{3}{9} = \frac{5}{9}$

$s_4 = \frac{1}{3} \cdot (\frac{5}{9} + 1) = \frac{5}{27} + \frac{9}{27} = \frac{14}{27}$

b) show $s_n > \frac{1}{2}, \forall n$

We will proceed with induction

Base case: n = 1

$s_1 = 1 > \frac{1}{2}$

Inductive hypothesis:

$s_n > \frac{1}{2}$

Inductive step:

$s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{s_n}{3} + \frac{1}{3} > \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$

Thus, using the fact that $s_n > \frac{1}{2}$, we can see:

$s_{n+1} > \frac{1}{2}$

c) show s_n is decreasing

we want to show

$s_{n+1} = \frac{1}{3}(s_n + 1) < s_n$

s_n is decreasing iff,

$s_n < 3s_n - 1$

$s_n > \frac{1}{2}$

Since this is a true statement which can be seen by part a, we can conclude that s_n is decreasing.

d) $\lim s_n$ exists because it is decreasing and bounded because $\frac{1}{2} < s_n < 1 \forall n$.

$\lim s_n = \frac{1}{2}$

Problem 10.11

$t_1 = 1, t_{n+1} = [1 - \frac{1}{4n^2}] * t_n, \forall n \geq 1$

Solution

a) $\lim t_n$ exists

First, we must see prove its a decreasing sequence.

This can be seen by looking at the ratio multiplying t_n

$r = 1 - \frac{1}{4n^2} < 1, \forall n > 0$

Since $r < 1$, any number $t > 0$

The following must hold:

$t * r > t * r^2 > t * r^3$

Since our sequence can be rewriten such that

$t_1 = t * r^0, t_2 = t * r, ...t_n = t * r^{n-1}$

$t_1 > t_2 > ...t_n$

Thus, t_n is a decreasing sequence.

Now, what bounds t_n?

Since t_n is decreasing, it is trivially upper bounded by its first element

$t_n \leq 1$

We can set the lower bound to 0.

This is because we start with an initial value $t_1 = 1 > 0$

The sequence itself is decreasing as we have proved before, but it will still be positive.

We know that from before, the sequence can be decomposed as a product of ratios r.

We can show that inf r = 1, and thus can never be negative.

$\lim\inf(1 - \frac{1}{4n^2}) = \lim_{N\to\infty} inf 1 - \frac{1}{4n^2} : n > N$

Looking at the first few terms of the sequence,

N = 0; $\inf\{1 - \frac{1}{4}, 1 - \frac{1}{16}, ...\} = \frac{3}{4}$

N = 1; $\inf\{1 - \frac{1}{16}, 1 - \frac{1}{36}, ...\} = \frac{15}{16}$

$\lim\inf r = 1 > 0$

Thus, our sequence will never multiply our base $t_1 = 1$ by a negative number, and thus the sequence t_n is strictly non-negative.

$t_n > 0$

We now have bounds for t_n

$0 < t_n \leq 1$

b) $\lim t_n = ?$

we can take the limit of an equivalent sequence:

$t_n = \prod_{i=1}^{n}(1 - \frac{1}{4i^2})^{i-1}$

$\lim \prod_{i=1}^{n}(1 - \frac{1}{4i^2})^{i-1}$

I have no idea what this is. I will guess that it is around 0.5 since it starts at 1, $\frac{3}{4}$...

and it is bounded (0,1)

Problem Squeeze Test

Let a_n, b_n, c_n be three sequences, such that $a_n \leq b_n \leq c_n$, and L = $\lim a_n = \lim c_n$, Show that L = $\lim b_n$

Solution We are given that, by definition of limits, $\exists \epsilon > 0 s.t. \forall n > N_0$

$|a_n - L| < \epsilon_a$

$L - \epsilon < a_n < L + \epsilon$

and

$\forall n > N_1$

$|c_n - L| < \epsilon$

$L - \epsilon < c_n < L + \epsilon$

Using what we are given:

$a_n \leq b_n \leq c_n$

implies, for $N = max(N_1, N_0)$

$L - \epsilon \leq b_n \leq L + \epsilon$

Which further implies for $n > N$

$|b_n - L| < \epsilon$

Which means

$\lim b_n = L$