Problem 10.6

(a) Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in N$.

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in N$?

Solution

a) We want to show that there exists some $\epsilon > 0$ s.t. $m > n > N$

$|s_m - s_n| < \epsilon$

We know that each successive term is within $\frac{1}{2^n}$ of the next one.

by triangle inequality,

$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} ... - s_{n+1} + s_{n+1} - s_n| < |s_m - s_{m-1}| + ... + |s_{n+1} - s_n|$

$|s_m - s_{m-1}| + ... + |s_{n+1} - s_n| < \frac{1}{2^{m-1}} + ... + \frac{1}{2^n} = \frac{1}{2^n}(\frac{1}{2^{m-n-1}} + ... + 1)$

We now recognise the right term as a geometric sum that is strictly less than its form that approaches infinity:

$\frac{1}{2^n}(\frac{1}{2^{m-n-1}} + ... + 1) < \frac{1}{2^n}(\sum_{i=0}^{\infty} \frac{1}{2^i})$

$\frac{1}{2^n}(\sum_{i=0}^{\infty} \frac{1}{2^i}) = \frac{1}{2^n} * 2 = \frac{1}{2^{n-1}}$

So we want to find an N such that $\frac{1}{2^{n-1}} < \epsilon$

$-(n-1) < log_2 \epsilon$

$n > -log_2 \epsilon + 1$

So setting $N = -log_2 \epsilon + 1$ as long as $0 < \epsilon < 2$

We have thus found an N s.t.

$|s_m - s_n| < \frac{1}{2^{n-1}} < \epsilon$

b) Since $\frac{1}{2^n} < \frac{1}{n}$ for n ¿ 1, we can not be sure that our same N works for a given ϵ.

Using our procedure from the first part, we can decompose again into a sum.

$|s_m - s_{m-1}| + ... + |s_{n+1} - s_n| < \frac{1}{m-1} + ... + \frac{1}{n} < \sum_{i=n}^{\infty}$

$\sum_{i=n}^{\infty}$ diverges, so s_n would not necessarily be Cauchy by our method.

Problem 11.2

Consider the sequences defined as follows:

$a_n = (-1)^n, b_n = \frac{1}{n}, c_n = n^2, d_n = \frac{6n+4}{7n-3}$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits.

(c) For each sequence, give its lim sup and lim inf.

(d) Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?

(e) Which of the sequences is bounded?

Solution

a) $a_n = (-1)^n : \{n | n \mod 2 = 1\}$

$b_n = \frac{1}{n} : \{n > 1\}$

$c_n = n^2 : \{n | n > 1\}$

$d_n = \frac{6n+4}{7n-3} : \{n | n > \frac{3}{7}\}$

b) $a_n = (-1)^n : S = \{-1, 1\}$

$b_n = \frac{1}{n} : S = \{0\}$

$c_n = n^2 : S = \{+\infty\}$

$d_n = \frac{6n+4}{7n-3} : S = \{\frac{6}{7}\}$

c) $a_n = (-1)^n : \limsup = 1, \liminf = -1$

$b_n = \frac{1}{n} : \limsup = 0, \liminf = 0$

$c_n = n^2 : \limsup = +\infty, \liminf = +\infty$

$d_n = \frac{6n+4}{7n-3} : \limsup = \frac{6}{7}, \liminf = \frac{6}{7}$

d $a_n = (-1)^n$: Diverges

$b_n = \frac{1}{n}$: Converges

$c_n = n^2$: Diverges to $+\infty$

$d_n = \frac{6n+4}{7n-3}$: Converges

e $a_n = (-1)^n$: Yes [-1,1]

$b_n = \frac{1}{n}$: Yes (0,1]

$c_n = n^2$, No

$d_n = \frac{6n+4}{7n-3}$ Yes $[\frac{10}{4}, \frac{6}{7})$

Problem 11.3

Repeat 11.2 for

$s_n = cos(\frac{n\pi}{3}), t_n = \frac{3}{4n+1}, u_n = -\frac{1}{2}^n, v_n = (-1)^n + \frac{1}{n}$

Solution

a) $s_n = cos(\frac{n\pi}{3}) : \{n|n \mod 6 = 0\}$

$t_n = \frac{3}{4n+1} : \{n > 0\}$

$u_n = -\frac{1}{2}^n : \{n|n > 0\}$

$v_n = (-1)^n + \frac{1}{n} : \{n|n \mod 2 = 0\}$

b) $s_n = cos(\frac{n\pi}{3}) : S = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$

$t_n = \frac{3}{4n+1} : S = \{\frac{3}{4}\}$

$u_n = -\frac{1}{2}^n : S = \{0\}$

$v_n = (-1)^n + \frac{1}{n} : S = \{-1, 1\}$

c) $s_n = cos(\frac{n\pi}{3}) : \limsup = 1, \liminf = -1$

$t_n = \frac{3}{4n+1} : \limsup = 0, \liminf = 0$

$u_n = -\frac{1}{2}^n : \limsup = 0, \liminf = 0$

$v_n = (-1)^n + \frac{1}{n} : \limsup = 1, \liminf = -1$

d $s_n = cos(\frac{n\pi}{3})$: Diverges

$t_n = \frac{3}{4n+1}$: Converges

$u_n = -\frac{1}{2}^n$: Converges

$v_n = (-1)^n + \frac{1}{n}$: Diverges

e $s_n = cos(\frac{n\pi}{3})$: Yes [-1,1]

$t_n = \frac{3}{4n+1}$: Yes $(0, \frac{3}{5}]$

$u_n = -\frac{1}{2}^n$, Yes $[-\frac{1}{2}, \frac{1}{4}]$

$v_n = (-1)^n + \frac{1}{n}$ Yes $[1, \frac{3}{2})$

Problem 11.5

Let (q_n) be an enumeration of all the rationals in the interval (0, 1].

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of lim sup q_n and lim inf q_n.

Solution

a) We know that by Denseness of rationals, every two real numbers a,b have a rational q in between them.

Let's assume that the two real numbers are actually just seperated by a 2ϵ

$a - \epsilon < q < a + \epsilon$

$|q - a| < \epsilon$

Since we did not specify r, we can arbitrarily pick any value a in in our bounds [0,1].

Thus, $S = [0, 1]$

b) For a bounded sequence,

$\limsup q_n = \sup q_n = \max s_n, \liminf q_n = \inf q_n = \min s_n$

The upper bound of q_n gives us $\max q_n = 1$, so $\limsup = 1$

$\liminf q_n = \lim inf\{q_n\}_{n=N}^{\infty}$

let $q_{n_k} = \{1, \frac{1}{2}...\} = \frac{1}{n}$ be a subsequence of q_n with the appropriate n_k

Our subsequence $q_{n_k} \leq q_n \forall n$:

$q_{n_k} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}...\}$

$q_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}...\}$

formally, we sort the sequences, we can write each entry as $q_{n_k} = \{\frac{1}{b}|b > 0\}, q_n = \{\frac{a}{b}|a, b > 0\}$

We know that by definition, $1 \leq a$, and thus $\frac{1}{b} \leq \frac{a}{b}$

Thus, $q_{n_k} \leq q_n$ and q_{n_k} is the smallest subsequence.

q_{n_k} is convergent and converges to 0, thus $\liminf q_{n_k} = 0$

The limit of the smallest subsequence is the liminf of the sequence.

$\liminf q_n = 0$

Problem Limsup

How would you explain 'what is limsup'? For example, you can say something about: What's the difference between limsup and sup? What is most counter-intuitive about limsup? Can you state some sentences that seems to be correct, but is actually wrong?

Solution I would explain limsup as the sup of a subset of the sequence that progressively shifts forward gradually. The difference between a limsup and a sup is that sup is a least upper bound of a sequence s_n whereas limsup is the limit of a subsequence of said sequence. Th e most counter-intuitive feature of limsup is that it is not always equal to the sup. This statement seems correct: limsup s = sup s. However, it is actually wrong since we can have a decreasing sequence where the sup of the set is not equal to the limsup.

$s_n = \frac{1}{n} = \{1, \frac{1}{2}, ...\}$

$\sup s_n = 1$

$\limsup s_n = \lim_{N\to\infty} \sup\{s_n | n > N\}$

$n = 1 = \sup\{1, \frac{1}{2}, ...\} = 1$

$n = 2 = \sup\{\frac{1}{2}, ...\} = \frac{1}{2}$

$\limsup s_n = 0$