Problem 12.10

Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$

Solution Suppose we had a bounded sequence $|s_n| \leq b; \forall n, b \in \mathbf{R}$

This implies that:

$\limsup |s_n| = \lim_{n \to \infty} sup\{A_n\}$

$A_n = \{|s_i| : i > n\}$

By definition of being bounded:

$sup\{A_n\} \leq b \forall n$

$\lim_{n \to \infty} sup\{A_n\} \leq b < \infty$

reverse direction:

If we had an $\limsup s_n < \infty$ implies s_n is a bounded sequence

let L be our what our limsup approaches.

$\lim_{n \to \infty} sup\{A_n\} \leq L < \infty$

limsup can be realized as actual sup for an ϵ

This, with an epsilon we can say that

$\sup\{A_n\} \leq L + \epsilon$

This can serve as a bound for s_n

$|s_n| \leq L$

Problem 12.12

$\sigma_n = \frac{1}{n}(s_1 + s_2 + ... + s_n)$.

Solution

1. Show $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$

   First inequality:

   $inf\{\sigma_n : n > N\} = inf\{\frac{1}{n}(s_1 + s_2 ... s_n) : n > N\}$

   The sum of n terms of a sequence can not be lower than the sum of the smallest element n times.

   $s_1 + s_2 ... s_n \geq n * \inf\{s_n\}$

   $\inf\{\frac{1}{n}(s_1 + s_2 ... s_n) : n > N\} \geq \inf\{s_n\}$

   This holds for all n

   $\liminf s_n \leq \liminf \sigma_n$

   Second inequality:

   $\limsup \sigma_n \leq \limsup s_n$

   The sum of n terms of a sequence can not be greater than the sum of the largest element n times.

   $s_1 + s_2 ... s_n \leq n * \sup\{s_n\}$

   $sup\{\frac{1}{n}(s_1 + s_2 ... s_n) : n > N\} \leq \sup\{s_n\}$

   This holds for all n

   $\limsup \sigma_n \leq \limsup s_n$

2. Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

   The limit existing implies the following:

   $\forall \epsilon > 0, \exists n > N s.t.$

   $|s_n - s| < \epsilon$

   By definition of σ_n :

   $\sigma_n = \frac{1}{n}(s_1 + s_2 + ... s_n)$

3. Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

   If we take the alternating sequence that doees not converge $s_n = -1^n$, the sequence itself does not converge but its average simplifies to a subsequence of $a_n = \frac{1}{n}$ which does converge

Problem 14.2

Which converge?

Solution

a) $\sum \frac{n-1}{n^2}$

This is a variant of the harmonic series because it can be reduced to $\frac{1}{n}$.

Diverges.

b) $\sum(-1)^n$

$(-1)^n$ is not decreasing nor is the limit of $(-1)^n$ convergent as a sequence.

Diverges

c) $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2}$

Integral test:

$\sum \frac{3}{n^2} \leq \int_1^\infty \frac{3}{n^2} = 3 * (\frac{1}{n})|_1^\infty = 3(1-0) = 3 < \infty$

By the integral test,

Converges

d) $\sum \frac{n^3}{3^n}$

Root test:

$|a_n|^{\frac{1}{n}} = |\frac{n^3}{3^n}|^{\frac{1}{n}} = \frac{n^{\frac{3}{n}}}{3}$

$\limsup n^{\frac{3}{n}} = \lim n^{\frac{3}{n}} = \lim n^{\frac{1}{n}} * \lim n^{\frac{1}{n}} * \lim n^{\frac{1}{n}} = 1 * 1 * 1 = 1$

$R = \frac{1}{3} < 1$

Converges

e) $\sum \frac{n^2}{n!}$

Ratio test:

$|\frac{a_{n+1}}{a_n}| = |\frac{(n+1)^2 n!}{(n+1)! n^2}| = |\frac{n+1}{n^2}|$

$\limsup |\frac{n+1}{n^2}| = \lim |\frac{n+1}{n^2}| = 0$

$R = 0 < \infty$

Converges

f) $\sum \frac{1}{n^n}$

Root test:

$|a_n|^{\frac{1}{n}} = |\frac{1^{\frac{1}{n}}}{n}| = |\frac{1}{n}|$

$\limsup \frac{1}{n} = 0$

$R = 0 < 1$

Converges

g) $\sum \frac{n}{2^n}$

Root test:

$|a_n|^{\frac{1}{n}} = |\frac{n}{2^n}|^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2}$

$\limsup n^{\frac{1}{n}} = 1$

$R = \frac{1}{2} < 1$

Converges

Problem 14.10

What is a series where the root test gives information whereas the ratio test does not.

Solution

$\sum 3^{-1^n - n}$

$|\frac{a_{n+1}}{a_n}|$:

for even n:

$= \frac{3^{-1-n-1}}{3^{1-n}} = \frac{1}{27}$

for odd n:

$= \frac{3^{1-n-1}}{3^{-1-n}} = 3$

$\frac{1}{27} = \liminf R < 1 < \limsup R = 3$

Ratio test is inconclusive

$|a_n|^{\frac{1}{n}}$:

for even n:

$3^{\frac{1}{n}-1}$

for odd n:

$3^{-\frac{1}{n}-1}$

$\lim 3^{\frac{1}{n}-1} = 3^{-\frac{1}{n}-1} = \frac{1}{3} < 1$

Converges by Root test

Problem 6

investigate convergence or divergence of $\sum a_n$

Solution

a) $a_n = \sqrt{n+1} - \sqrt{n}$

Diverges

Writing out the first terms of the series, we can identify it as a telescoping series.

$(\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) ... (\sqrt{N+1} - \sqrt{N})$

Which simplifies to the following via a rearrangement:

$-\sqrt{1} + \sqrt{N+1}$

The sum diverges to infinity as N approaches infinity, so the series itself does not converge.

b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

rewrite the series:

$\frac{\sqrt{n+1} - \sqrt{n}}{n} * \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{n(\sqrt{n+1} + \sqrt{n})} < \frac{1}{n*2\sqrt{n}} = \frac{1}{2*n^{\frac{3}{2}}}$

The right of the inequality is a p-series with p ¿ 1, which converges by the following integral test:

$\int_1^\infty \frac{1}{2*n^{\frac{3}{2}}} dn = \frac{-1}{n^{\frac{1}{2}}}|_1^\infty = -0 + 1 = 1$

Thus, the series is strictly less than a convergent series and by the comparison test the series a_n converges.

c) $a_n = (\sqrt[n]{n} - 1)^n$

Root test:

$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n} - 1$

$\limsup_{n\to\infty} \sqrt[n]{n} - 1 = (\limsup_{n\to\infty} \sqrt[n]{n}) - 1 = 1 - 1 = 0$

$\alpha = 0 < 1$

Thus, by the root test the series converges.

d) $a_n = \frac{1}{1+z^n}$; for complex values of z

Problem 7

Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{a_n}{n}$ if $a_n \geq 0$

Solution If $a_n > 0$, then $a_n > \frac{a_n}{n} \forall n$

This implies:

$\sum a_n > \sum \frac{a_n}{n}$

By the comparison test, since $\sum a_n$ converges, $\sum |a_n|$ converges,

$\sum \frac{a_n}{n} < \sum |a_n|$ implies

$\frac{a_n}{n}$ converges

Problem 9

Find the radius of convergence of each of the following power series

Solution

a) $\sum n^3 z^n$

$\alpha = \limsup |n^3|^{\frac{1}{n}} = |n^{\frac{3}{n}}| = |\sqrt[n]{n}^3| = |\sqrt[n]{n}| * |\sqrt[n]{n}| * |\sqrt[n]{n}|$

$|\sqrt[n]{n}|$ is convergent to 1, thus the following is true:

$\alpha = 1 * 1 * 1 = 1$

$R = 1$

b) $\sum \frac{2^n}{n!} z^n$

$\alpha = \limsup |\frac{2^n}{n!}|^{\frac{1}{n}} = \limsup |2| * |\frac{1}{n!^{\frac{1}{n}}}| = 2 * \limsup |\frac{1}{n!^{\frac{1}{n}}}|$

$\lim n!^{\frac{1}{n}} = \infty$

$\limsup |\frac{1}{n!^{\frac{1}{n}}}| = 0$

$\alpha = 2 * 0 = 0$

$R = \infty$

c) $\sum \frac{2^n}{n^2} z^n$

$\alpha = \limsup |\frac{2^n}{n^2}|^{\frac{1}{n}} = \limsup |2| * |\frac{1}{n^{\frac{2}{n}}}| = 2 * 1 = 2$

$R = \frac{1}{2}$

d) $\sum \frac{n^3}{3^n} z^n$

$\alpha = \limsup |\frac{n^3}{3^n}|^{\frac{1}{n}} = \limsup |\frac{1}{3}| * |n^{\frac{3}{n}}| = \frac{1}{3} * 1 * 1 * 1 = \frac{1}{3}$

$R = 3$

Problem 11

Suppose $a_n > 0, s_n = a_1 + ... + a_n, \sum a_n$ diverges

Solution

a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges

We can use the contrapositive of the sanity check.

Since a_n is positive and divergent,

$\lim_{n->\infty} \frac{a_n}{1+a_n} = 1$

Since the limit of the terms is not 0, then the series must diverge.

b) Prove that

$\frac{a_{N+1}}{s_{N+1}} + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$

and deduce that $\sum \frac{a_n}{s_n}$ diverges

c) Prove that

$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

d) What can be said about

$\sum \frac{a_n}{1+n*a_n}$

and

$\sum \frac{a_n}{1+n^2*a_n}$