Problem 13.3

Let B be the set of all bounded sequences $x = (x_1, x_2, ...)$, and define $d(x, y) = \sup\{|x_j - y_j| : j = 1, 2, ...\}$.

Solution

a) Show d is a metric:

$d(x, x) = 0$

Proof:

Fact: $x_j - x_j = 0$

$\sup\{|x_j - x_j| : j = 1, 2, ...\} = \sup\{0 : j = 1, 2, ...\} = 0$

□

$d(x, y) = d(y, x)$ Proof: $\sup\{|x_j - y_j| : j = 1, 2, ...\} = \sup\{|-(y_j - x_j)| : j = 1, 2, ...\}$

Using the following fact about absolute values:

$|-1 * x| = |x|$

$\sup\{|-(y_j - x_j)| : j = 1, 2, ...\} = \sup\{|y_j - x_j| : j = 1, 2, ...\}$

□

$d(x, z) \leq d(x, y) + d(y, z)$

Proof:

Triangle inequality

$|x_j - z_j| = |x_j - y_j + y_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$

Thus, applying the inequality to every entry j:

$\sup\{|x_j - z_j| : j = 1, 2, ...\} \leq \sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, ...\}$

Lemma proved in past hw: $\sup(a + b) \leq \sup(a) + \sup(b)$

$\sup\{|x_j - y_j| + |y_j - z_j| : j = 1, 2, ...\} \leq \sup\{|x_j - y_j| : j = 1, 2, ...\} + \sup\{|y_j - z_j| : j = 1, 2, ...\}$

□

b) No, because this summation does not necessarily converge. Thus, we can not guarantee the fulfillment of distance metrics.

Take x = [1,1,1...], y = [0,0,0...], z = [2,2,2...]

$d(x, z) \leq d(x, y) + d(y, z)$

This inequality can not hold.

Problem 13.5

Verify DeMorgan's Laws for sets

Show that the intersection of any collection of closed sets is closed

Solution

a) Let $s \in \bigcap\{S/U : U \in \mathcal{U}\}$ be any element s that is in S but is not inside any set U in $\mathcal{U}$

This also means that the element s is not in the union of all of the sets U in $\mathcal{U}$

Thus,

$s \in S/\bigcup\{U : U \in \mathcal{U}\}$

Thus, since each s belongs to both sets we can establish that these two sets are equal.

$\bigcap\{S/U : U \in \mathcal{U}\} = S/\bigcup\{U :\in \mathcal{U}\}$

b) The intersection of any closed sets U of a collection of sets $\mathcal{U}$ of a space S:

Define U as an open set $\in \mathcal{U}$

Using the fact that any arbitrary union of open sets is another open set:

$\bigcup\{U : U \in \mathcal{U}\}$ is an open set

The compliment of an open set is a closed set:

$S/\bigcup\{U : U \in \mathcal{U}\}$ is closed

Using part A, we know an equivalent definition for this open set as follows:

$\bigcap\{S/U : U \in \mathcal{U}\}$ is closed

S/U is also a closed set by the above compliment property, making this an intersection of closed sets.

Thus, we have show that the intersection of closed sets is also a closed set.

Problem 13.7

Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

Solution Let us have an arbitrary open set in $S \in \mathbb{R}$

We know that any arbitrary union of open sets will also be open.

We also know that by definition of open, there exists an open ball around any point p $\in S$ with radius r:$B_r(p)$

We can place an infinite amount of these open balls within the open set and take the union of them.

The arbitrary union of open sets (open balls are open sets) is an open set.

$\bigcup\{B_r(p) : p \in S\}$ is open and given an infinite amount of these balls we will have S.

However, each ball is not guaranteed to be disjoint, so we can merge any open sets that overlap until the sets are disjoint, thus completing our proof.

Problem 4

Recall that in class, given (X, d) a metric space, and S a subset of X, We defined the closure of S to be:

$\bar{S} = \{p \in X \mid \text{there is a subsequence } (p_n) \in S \text{ that converge to } p\}$

Prove that taking closure again won't make it any bigger, i.e, if $S_1 = \bar{S}$, and $S_2 = \bar{S}_1$, then $S_1 = S_2$.

Solution If we are to take the closure $S_1 = \bar{S}$,

Since by definition, taking a closure of any set results in a subset,

$S_1 = \{p \in X : \exists (p_n) \in S, p_n \to p\}$

$S_2 = \bar{S}_1$

$S_2 \subseteq S_1$

By definition of closure,

$S_2 = \{q \in X : \exists (q_n) \in S_1, q_n \to q\}$

We want to show that $S_1 \subseteq S_2$

or, equivalently, every point $p \in S_1$ can be found in S_2

Since S_2 contains points q that have sequences converging to them, we can set up a diagonalization argument.

Let q_{ij} be a sequence$\in S_2$ that approaches p_i

$q_{11}, q_{12}, q_{13} - > p_1$

$q_{21}, q_{22}, q_{23} - > p_2$

...

$q_{n1}, q_{n2}, q_{n3} - > p_n$

We can arbitrarily reconstruct the sequence that approaches $p \in S_1$, thus every point in S_1 is found in S_2 and $S_1 = S_2$

Problem 5

Prove that $\bar{S}$ is the intersection of all closed subsets in X that contains S. (you may assume result in 4, namely, $\bar{S}$ is closed)

Solution We need to first define the intersection of all closed subsets in X that contain S.

Recall: $\bar{S} \subseteq S$

Let U be a closed subset in X that contains S

$\bar{S} \subseteq S \subseteq U$

Let the following be the intersection of all closed subsets of X.

$U' = \bigcap\{U_\alpha : \forall\alpha \in I\}$

Since $\bar{S}$ is closed, it is also included in this union

Since an intersection of a set can be at most exactly as big as any member of the set, then

$U' \subseteq \bar{S}$

Since closures can not decrease in size, $\bigcap \bar{S} = \bar{S} \subseteq U'$

Thus, $\bar{S} = U'$

so the closure of S is the intersection of all closed subsets containing S.