Problem 1

In class, we proved that $[0, 1]$ is sequentially compact, can you prove that $[0, 1]^2 \in \mathbb{R}$ is sequentially compact? (In general, if metric space X and Y are sequentially compact, we can show that X $\times$ Y is sequentially compact.

Solution We know from lecture that $[0, 1]$ is compact because it is a closed and bounded. By the Bolzano Weierstrass theorem, this implies that each point $p \in [0, 1]$ is a convergent sub-sequence (p_n).

We want to show that a similar notion of sequential compactness holds in 2 dimensions.

Take any point $[i, j] \in [0, 1]^2$

We know that individually, there must be convergent sub-sequences that approach any point in [0,1].

Define the sub-sequences in singular dimensions

$\exists N_i s.t. \forall n > N_i, d([u_n, 0], [u, 0]) < \frac{\epsilon}{2}$

$\exists N_j s.t. \forall n > N_j, d([0, v_n], [0, v]) < \frac{\epsilon}{2}$

If we take $\bar{N} = \max(N_i, N_j)$, then the following pair must hold.

$\forall n > \bar{N}, d([u_n, v_n], [u, v]) < d([u_n, 0], [u, 0]) + d([0, v_n], [0, v]) = \epsilon$

Then, for any point in $[0, 1]^2$

$[u_n, v_n] \to [u, v]$

Problem 2

Let E be the set of points $x \in [0, 1]$ whose decimal expansion consist of only 4 and 7 (e.g. 0.4747744 is allowed), is E countable? is E compact?

Solution Assume that E is finite, then we can enumerate every number $p \in E$ in the following way.

$p_i = \sum_{i=1}^{n} \frac{4}{10^i}$

$p_1 = 0.4$

$p_2 = 0.44$

If we have enumerated all finitely many n of them, $p_1 ... p_n$

We can always construct one that is not enumerated by taking $p_{n+1} \in E$

Thus, E is not countable.

For compactness,

We can show that through the proving E is closed and using Heine-Borel.

We proceed by observing that the complement E^c is an open set.

E^c is the set of all numbers that are not purely digits of 4's and 7's

Prove E^c is open.

We can define a valid definition of E^c:

Let us first define a useful set of open intervals that 'fills in the gaps' of E.

$U = \bigcup_i U_i$

order the numbers in $\tilde{E}$:

$(0.4, 0.44, 0.\bar{4}, ...0.\bar{7}$

in between every element of E, there is a open interval we can place such that no element of E is withing U_i. Since we can have an infinite amount of these intervals, we can account for any granuality of E.

$U_i = (\tilde{E}_i, \tilde{E}_{i+1})$

$E^c = (0, 0.4) \bigcup (0.7, 1) \bigcup U$ is an open interval

Thus, E is closed.

E is compact.

Problem 3

Let $A_1, A_2, \cdots$ be subset of a metric space. If $B = \cup_i A_i$, then $\bar{B} \supset \cup_i \bar{A}_i$. Is it possible that this inclusion is an strict inclusion?

Solution

Take A to be the following set of covers

$A_i = (1/i, 1)$

taking the infinite union of all the subsets, we will construct B as the following:

$B = \bigcup_i A_i = (0, 1)$

Taking the closure of this infinite set B,

$\bar{B} = [0, 1]$

This closure contains right end point of 0,

However, there can not be a closure of A that can contain 0.

$\forall i, \{0\} \bigcap [1/i, 1] = \emptyset$

We have a point in $\bar{B}$ that is not in $\bigcup_i A_i = (0, 1)$, thus showing that the subset is strict in this case.

Problem 4

Last time, we showed that any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals. Here is a claim and argument about closed set: every closed subset of $\mathbb{R}$ is a countable union of closed intervals. Because every closed set is the complement of an open set, and adjacent open intervals sandwich a closed interval. Can you see where the argument is wrong? Can you give an example of a closed set which is not a countable union of closed intervals? (here countable include countably infinite and finite)

Solution Take the set of real numbers R, which is a subset of R. We know that by definition, R itself is closed in R trivially because it contains all the limit points in R. However, we also know that R is not countable.

No matter how many closed intervals we use to try and cover R, we can not fully reconstruct R using a union of finite closed intervals.

Take any finite set of intervals U. $\forall i, U_i = [i, i]$

The union of all the intervals is just the largest interval since the U_{i+1} interval is a strict subset of the U_i

$\bigcup U_i = U_{max(i)}$

There exists a real number $i + \epsilon$ that exists outside of this union. Thus, we can not cover R using a finite set of closed intervals.