Problem 1

If X and Y are open cover compact, can you prove that $X \times Y$ is open cover compact? (try to do it directly, without using the equivalence between open cover compact and sequential compact)

Solution Let us define what X and Y being open cover compact tells us.

Any open cover of X and Y must admit a finite subcover.

Let $X \subseteq \bigcup_i U_i, Y \subseteq \bigcup_j V_j$

We can write a very large cover of this space by taking increasingly large open balls over one axis.

$X \times Y \subseteq \{(a, b) | a \in X, b \in Y\}$

Using the fact that Y is compact, we can rewrite the above arbitrary cover using a finite cover. For some finitely large $1 < j < J$,

$X \times Y \subseteq \bigcup_a \{(a, v) | v \in \bigcup_1^J V_j\}, a \in X$

Since a is any arbitrary element of X, by X being compact we know that we can admit a finite open cover that contains all elements a in X. Thus, the largest that $\bigcup_a \{(a, v) | v \in \bigcup_1^J V_j\} a \in X$ is still finite. For a finitely large $1 < i < I$

$\bigcup_a \{(a, v) | v \in \bigcup_1^J V_j\} \subseteq \bigcup_{i,j} \{(u, v) | u \in \bigcup_1^I U_i, v \in \bigcup_1^J V_j\}$

$X \times Y \subseteq \bigcup_{i,j} \{(u, v) | u \in \bigcup_1^I U_i, v \in \bigcup_1^J V_j\}$

since both $\bigcup_1^I U_i, \bigcup_1^J V_j$ are finite unions of sets, then their union is also finite and thus any arbitrary cover of $X \times Y$ is a subcover of a finite cover which makes them also finite covers.

Problem 2

Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \in X$ be a subset. Decide if the followings are true or not. If true, give an argument, if false, give a counter-example.

if A is open, then $f(A)$ is open

if A is closed, then $f(A)$ is closed.

if A is bounded, then $f(A)$ is bounded.

if A is compact, then $f(A)$ is compact.

if A is connected, then $f(A)$ is connected.

Solution

1. if A is open, then $f(A)$ is open

   False: $A = \mathbb{R}, f(x) = c$

   If we map all the real numbers to a constant, then this is not open in Y since it is a single point and any ball we draw around c will have points that are not in f(A) but are in Y.

2. if A is closed, then $f(A)$ is closed.

   False: $A = [0, 1], f(x) = \frac{1}{n}$

   f(x)'s image $(-\infty, 1]$ does not contain the left limit point of f(x).

3. if A is bounded, then $f(A)$ is bounded.

   False: $A = [0, 1], f(x) = \frac{1}{n}$

   f(x)'s image $(-\infty, 1]$ does not have a lower bound.

4. if A is compact, then $f(A)$ is compact.

   True: Take A that is sequentially compact and thus every sequence has a convergent subsequence.

   Take the elements $a_n \to a$. Since the sequence converges to a, there are infinitely many a_n that are within ϵ of a.

   By continuity, $f(a)$ must have infinitely many points $f(a_n)$ that are within δ of $f(a)$. This creates a convergent sequence in $f(a)$, thus showing that $f(A)$ is sequentially compact.

5. if A is connected, then $f(A)$ is connected.

   True: Let us assume that A is connected but f(A) is not connected.

   This means that f(A) can be written as the disjoint union $f(A_L) \bigsqcup f(A_R)$.

   Pulling $f(A_L), f(A_R)$ through the inverse function, $A = A_L \bigsqcup A_R$. This means that A is not connected. Thus, we have a contradiction.

Problem 3

Prove that, there is not continuous map $f : [0, 1] \to \mathbb{R}$, such that f is surjective. (there is a surjective map from $(0, 1) \to \mathbb{R}$ though)

Solution Surjective: onto or maps to all elements in $\mathbb{R}$

To prove this, we can just show that there exists an element in $\mathbb{R}$ that is not mapped to from $[0, 1]$ by f.

A key thing to notice is that the set that the pre-image belongs to is bounded. By this bound, we know that there does exists a way to construct some kind of element that lies outside of our pre-image. Let $p = 1 + \delta$

Since the distance between p and the right endpoint is within δ, continuity requires us to have a point in our image such that:

$d(f(x), f(p)) \leq \epsilon$

However, f(p) can not be in our image if it is outside of our pre-image.

We can not take the distance between two points that are not both in our image. Thus, this leads to a contradiction.