Problem 1

Let $f_n(x) = \frac{n+\sin x}{2n+\cos n^2 x}$, show that f_n converges uniformly on $\mathbb{R}$.

Solution

Definition of uniform convergence: For every $\epsilon > 0$, there exists a $n \geq N$ such that:

$|f_n(x) - f(x)| \leq \epsilon$

Let $f(x) = \frac{1}{2}$

Taking a closer look at, $f_n(x)$, we can realize that it is upper bounded by a much simpler function

$\frac{n+\sin x}{2n+\cos n^2 x} \leq \frac{n+1}{2n}$

Which leads us to the following inequality for ϵ

$|f_n(x) - f(x)| = |\frac{n+1}{2n} - \frac{1}{2}| = |\frac{1}{2n}| \leq \epsilon$

Choosing $N = \frac{1}{2\epsilon}$ will satisfy the definition of uniform convergence.

Problem 2

Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$.

Show that the series is continuous on $[-1, 1]$ if $\sum_n |a_n| < \infty$.

Prove that $\sum_{n=1}^{\infty} n^{-2} x^n$ is continuous on $[-1, 1]$.

Solution

1. We know that $\sum |a_n| < \infty$, so we can use the Weierstrass M-test.

   $|f_n(x)| = |a_n x^n| = |a_n| * |x^n| \leq |a_n| \forall x \leq 1$

   Thus, $f(x) = \sum f_n(x)$ converges uniformly.

   By theorem 7.12 in Rudin, this is sufficient to show that f is continuous.

2. The above case covers the case for $a_n = n^{-2}$ because $\sum n^{-2}$ is convergent.

   Intergral test:

   $\int_1^{\infty} n^{-2} dx = -n^{-1}|_1^{\infty} = (\frac{-1}{\infty} + \frac{1}{1}) = 1$

Problem 3

Show that $f(x) = \sum_n x^n$ represent a continuous function on $(-1, 1)$, but the convergence is not uniform.

(Hint: to show that $f(x)$ on $(-1, 1)$ is continuous, you only need to show that for any $0 < a < 1$, we have uniform convergence on $[-a, a]$. Use Weierstrass M-test.)

Solution We want to prove continuity, but show that the convergence is not uniform.

For just continuity, using the Weierstrass M-test,

Verify that $\sum |a^n| = \frac{1}{1-a} < \infty$

It can be seen that $|x^n| \leq a^n$ for all $a \in (0, 1), x \in (-a, a)$, and thus $f(x)$ is a continuous function on $(-1, 1)$ since it converges uniformly $(-a, a)$ for any aforementioned value of a.

Definition of uniform convergence:

for every $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| \leq \epsilon, \forall x$

$f_n(x) = \sum_{i=1}^{n} x^i, f(x) = \sum_{n=1}^{\infty} x^n$

$|f_n(x) - f(x)| = |\frac{x(1-x^n)}{1-x} - \frac{x}{1-x}| = |\frac{x^{n+1}}{x+1}| \leq \epsilon$

However, if we set $\epsilon = 0.01$, then there exists a value x where the above is false.

$x = 0.99, N = 1$

Examples: $|\frac{0.99^2}{1.99}| > 0.01$

This directly shows that x_n is not uniformly convergent