Problem 1

Read Ross p257, Example 3 about smooth interpolation between 0 for $x \leq 0$ and $e^{-1/x}$ for $x > 0$. Construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$, and $f(x) \in [0, 1]$ when $x \in (0, 1)$.

Solution So we want a function that fulfills the above requirements of being 0 for non-negative values of x, 1 for values of x greater than 1, and is smooth.

To be smooth, it's derivative must exist for all values of X.

Trivially, f is differentiable and smooth for values $x < 0$ and $x > 1$. This is because a constant function's derivative is identically 0.

So, we also want our derivative to have a value of 0 as it approaches 0 from the left and 1 from the right.

$lim_{x\to 0+} \frac{f^{(n)}(x)-f^{(n)}(0)}{x-0} = 0$

$lim_{x\to 1-} \frac{f^{(n)}(x)-f^{(n)}(1)}{x-1} = 0$

After thinking for a long time, we can try a "middle function" of $e^{1-\frac{1}{x}}$

Verify that $f(1) = 1, f(0) = 0$

Let's verify the limit definition and show that the function is still smooth:

First, let's analyze the derivatives $f^{(n)}(x)$

We claim that for $f(x) = \frac{1}{x}e^{1-\frac{1}{x}}$, there exists a polynomial $p_n(\frac{1}{x})$ such that

$f^{(n)}(x) = e^{1-\frac{1}{x}} * p_n(\frac{1}{x})$

We see it is true for n = 1, $p_n(t) = t^3 - t^2$. By a similar process to the book, we see that $p_n(x)$ is a degree 2n polynomial.

Now, we proceed with the first requirement.

By a similar process to the book, at point 0 we can use the proven statement that $\lim_{x\to\infty} \frac{1}{x^k}e^{-x} = 0$

$\lim_{x\to 0+} \frac{f^{(n)}(x)-f^{(n)}(0)}{x-0} = \lim_{x\to 0+} \frac{1}{x} * \frac{e^{1-\frac{1}{x}}}{x} * p_n(\frac{1}{x}) = e * \lim_{x\to 0+} e^{-\frac{1}{x}} * p_n(\frac{1}{x}) = 0$

Now for the second requirement at 1. This is newer but we can try.

$\lim_{x\to 1-} \frac{f^{(n)}(x)-f^{(n)}(1)}{x-1} = \frac{1}{x-1}(\frac{e^{1-\frac{1}{x}}}{x}p_n(\frac{1}{x}) - 1)$

Since $\frac{1}{x-1}$ approaches ∞, we can use L'hopital's rule.

$= \lim_{x\to 1-} \frac{e^{1-\frac{1}{x}}}{x}p_{n+1}(\frac{1}{x})$

After realizing we have made a function that is not smooth, we consult our peers for a different candidate function.

Apparently taking the function provided by Rudin $f(x) = e^{-\frac{1}{x}}; x > 0$ and doing the following works:

$\frac{f(x)}{f(x)+f(1-x)}$

They also justify that the function is smooth since it is a composition of other smooth functions.

Problem 2

Rudin Ch 5, Ex 4 (hint: apply Rolle mean value theorem to the primitive)

Solution We want to prove that if $C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$

Then,

$f(x) = C_0 + C_1x + ...C_nx^n = 0$ has at least 1 real root between 0 and 1.

Well, let us construct an anti-derivative for the above equation. This function, we will call g, must equal 0 at x = 0 and x = 1.

$g(x) = C_0x + \frac{C_1x^2}{2} + ... + \frac{C_{n-1}x^n}{n} + \frac{C_nx^{n+1}}{n+1}$

For x = 0, we can see by substitution we will have a sum of all 0 terms since all terms have 0.

For x = 1, we reconstruct the following:

$C_0 + \frac{C_1}{2} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1}$

We are also given that the above also equals 0.

Now, by Rolle's theorem we can say that the derivative of the function g has a value c where its derivative, $g'(c)$, equals 0

We know that if we take the derivative, we obtain f(x).

$g'(x) = f(x) = C_0 + C_1x + ...C_nx^n$

Thus, using a = 0, b = 1, we have thus shown that there must exist a c s.t. $f(c) = 0 \in [0, 1]$

Problem 3

Rudin Ch 5, Ex 8 (ignore the part about vector valued function. Hint, use mean value theorem to replace the difference quotient by a differential)

Solution We are given that f' is continuous on [a,b]. For a $\epsilon > 0, \exists\delta > 0$ s.t

$0 < |t - x| < \delta, a \leq x \leq b, a \leq t \leq b$

and we want to show

$|\frac{f(t)-f(x)}{t-x} - f'(x)| \leq \epsilon$

Using the mean value theorem, we know there exists a $c \in [t, x]$, wlog assume $t < x$ such that

$f'(c) = \frac{f(t)-f(x)}{t-x}$

We also know that since $c \in [t, x]$ and $0 < |t - x| < \delta$,

$|c - x| < \delta$

Using continuity of f', we know the corresponding images of x and c are within an epsilon distance of each other.

By continuity of f'

$|c - x| < \delta \to |f'(c) - f'(x)| \leq \epsilon$

Problem 4

Rudin Ch 5, Ex 18 (alternative form for Taylor theorem)

Solution From differentiation, we get the following formula by induction:

$f^{n-1}(t) = (t-\beta)Q^{n-1}(t) + (n-1)Q^{n-2}(t)$

With the definition for the Taylor polynomial P around the point α:

$P_\alpha(x) = \sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(x-\alpha)^k + f(\alpha)$

Sub in:

$P_\alpha(\beta) = \sum_{k=0}^{n-1} \frac{(\alpha-\beta)Q^k(\alpha)+kQ^{k-1}(\alpha)}{k!}(\beta-\alpha)^k + f(\alpha)$

Split the parts of the sum:

$= \sum_{k=0}^{n-1} \frac{(\beta-\alpha)^k Q^{k-1}(\alpha)}{(k-1)!} - \frac{(\beta-\alpha)^{k+1}Q^k(\alpha)}{k!} + f(\alpha)$

The second term will always be canceled out by the previous first term, leaving us with the following by the end:

$= (\beta-\alpha)Q(a) + \frac{(\beta-\alpha)^n Q^{n-1}(\alpha)}{(n-1)!} + f(\alpha)$

Simplify:

$P_\alpha(\beta) = f(\beta) - \frac{(\beta-\alpha)^n Q^{n-1}(\alpha)}{(n-1)!}$

Thus:

$f(\beta) = P_\alpha(\beta) + \frac{(\beta-\alpha)^n Q^{n-1}(\alpha)}{(n-1)!}$

Problem 4

Rudin Ch 5, Ex 22

Solution

1. Suppose there were two such fixed points a,b

   Since f is differentiable, by the mean value theorem there must exist a $c \in [a, b]$

   such that

   $f'(c) = \frac{f(a)-f(b)}{a-b} = \frac{a-b}{a-b} = 1$

   However, this contradicts our assumption. Thus, there can be no more than 1 fixed point on f.

2. f has no fixed point because every item t is strictly greater than its corresponding fixed point f(t).

   For a point t to be a fixed point,

   $f(t) = t + \frac{1}{1-e^t} = t$

   which implies $\frac{1}{1+e^t} = 0$

   but $\frac{1}{1+e^t}$ never realizes 0.

3. We know that for f to have a fixed point,$f(x) = x$

   We proceed by first showing that x_n converges because it is cauchy.

   We are given that $|f'(t)| \leq A < 1$

   Taking one step in our algorithm, we see that we can bound the distance between our (x_n, x_{n+1})

   $|f'(t)| = |\frac{f(x_n)-f(x_{n+1})}{x_n-x_{n+1}}| < 1$

   $f(x_n) - f(x_{n+1}) \leq x_n - x_{n+1}$

   We know that the distance between each pair in each subsequent step of our algorithm must decrease by induction. x_n is cauchy.

   We will call the convergent point x.

   $\lim x_n = x$

   Since $f(x_n) = x_{n+1}$, $f(x_n)$ is a subsequence because it is just the continuation of x_n which must also converge to x.

   $\lim f(x_n) = \lim x_{n+1} = x$

4. Starting a $(x_1, f(x_1)) = (x_1, x_2)$

   moving from $(x_1, x_2) \to (x_2, x_2)$ is fine since

   $\frac{f(x_2)-f(x_2)}{x_2-x_2} = 0 < 1$

   (x_2, x_2) onwards is similar