10.9) a) $s_2 = \frac{1}{2}$, $s_3 = \frac{2}{12} = \frac{1}{6}$, $s_4 = \frac{3}{144} = \frac{1}{32}$

b) s_n is monotonically decreasing, because squaring previous value less than 1, and is bounded below because always positive so limit exists.

c) $0 \leq s_n \leq \frac{1}{2}^{n} \cdot 2$ since always positive and at each step multiplies by less than or equal to $\frac{1}{2}$ and $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} 2^{-n+1} = 0$

10.10) a) $s_2 = \frac{2}{3}$ $s_3 = \frac{5}{9}$, $s_4 = \frac{14}{27}$

b) $s_2 = \frac{2}{3} > \frac{1}{2}$

if $s_n > \frac{1}{2}$, then $s_{n+1} > \frac{1}{3}(\frac{1}{2}+1) = \frac{1}{2}$ so $s_{n+1} > \frac{1}{2}$

c) Since $s_{n+1} - s_n = -\frac{2}{3} s_n + \frac{1}{3}$ and $s_n > \frac{1}{2}$ $\forall n$

$s_{n+1} - s_n < 0$ so decreasing

d) Since lower bound and monotonic decreasing, limit exists.

$\forall \varepsilon > 0$, $\exists N$ st $\forall n > N$

$s_{n+1} < \frac{1}{2} + \varepsilon$

$\frac{1}{3} s_n < \frac{1}{6} + \varepsilon$

$s_n < \frac{1}{2} + 3\varepsilon$

So just multiply ε by three until you get to $\varepsilon \approx 1$ then take N to be that many times multiplying by 3

$$\varepsilon \cdot 3^N = 1$$

$$N = \log_3\left(\frac{1}{\varepsilon}\right)$$

10.11) a) Since multiplying by pos fraction $t_n > 0$ but decreasing so limit must exist.

b) This is a lil tricky $t_2 = \frac{3}{4}$ $t_3 = \frac{45}{64}$

Maybe around $\frac{2}{3}$?

Python indicates around .637

$\forall \varepsilon > 0$ let $N = \max(N_1, N_2)$ where N_1 is such that $\forall n > N_1$ $|a_n - L| < \varepsilon$ & $\forall n > N_2$ st $\forall n > N_2$ $|c_n - L| < \varepsilon$.

$\therefore \forall n > N$, $a_n - L \le b_n - L \le c_n - L$

if $b_n - L > 0$, then $|b_n - L| \le |c_n - L| < \varepsilon$
if $b_n - L < 0$, then $|b_n - L| < \varepsilon$
thus $\lim b_n = L$

Discussion Questions

9.9) a) $\forall \varepsilon > 0$, $\exists N_1$ st $\forall n > N_1$ $s_n > \varepsilon$, $\forall n > \max(N_0, N_1)$ $t_n > \varepsilon$
b) $\forall \varepsilon < 0$, $\exists N_1$ st $\forall n > N_1$ $s_n < \varepsilon$, $\forall n > \max(N_0, N_1)$ $t_n < \varepsilon$
c) Assume $\lim s_n > \lim t_n$
Take $\varepsilon = (\lim s_n - \lim t_n)/2$
Then there exists n_1 st $s_n > t_n$ which is contrad

9.15) For $n > a$, $\frac{a^{n+1}}{(n+1)!} = \frac{a^n}{n!} \cdot \frac{a}{n+1}$

$\lim \frac{a}{n+1} = 0$, so multiplying by near zero at each step, brings it to near 0.

10.7) WLOG, consider subset as $(0,1)$ (can multiply by $\frac{1}{\sup(S) - \inf(S)}$)
Consider seq $1 - \frac{1}{2n}$ for $n \ge 1$.
It approaches 1

10.8) $\sigma_{n+1} - \sigma_n = \frac{s_{n+1} + \dots + s_1}{n+1} - \frac{s_n + \dots s_1}{n}$

WTS $n s_{n+1} + n(s_n + \dots + s_1) > n(s_n + \dots + s_1) + (s_n + \dots + s_1)$
$n s_{n+1} > s_n + \dots + s_1$
$n s_{n+1} > n s_n = \underbrace{s_n + \dots + s_n}_{n} > s_n + \dots + s_1$