13.3, 13.5, 13.7

13.3) a) $d(x,y)=0 \Leftrightarrow \sup\{|x_i-y_i| : i=1,2,\ldots\}=0 \Leftrightarrow |x_i-y_i|=0 \ \forall\, i=1,2,\ldots$

$\Leftrightarrow x=y$ ✓

$d(y,x)=\sup\{|y_i-x_i| : i=1,2,\ldots\}=\sup\{|x_i-y_i| : i=1,2,\ldots\}=d(x,y)$ ✓

$d(x,y)=\sup\{|x_i-y_i| : i=1,2,\ldots\}=\sup\{K : K\geq 0\}\geq 0$ ✓

$d(x,y)+d(y,z)=\sup\{|x_i-y_i|\}+\sup\{|y_j-z_j|\}$

if sup is for $i=j$ then $d(x,y)+d(y,z)=d(x,z)$
else $d(x,y)+d(y,z)>d(x,z)$

b) No. Metrics must be finite but if $x=(1,1,1,\ldots)$
and $y=(0,0,0,\ldots)$ then the sum is infinite.

13.5) a) $x\in\bigcap\{S\setminus U : U\in\mathcal{U}\}\Leftrightarrow \forall\, U\in\mathcal{U},\ x\in S\setminus U$
$\Leftrightarrow x\in S$ and $x\notin U\ \forall\, U\in\mathcal{U}$
$\Leftrightarrow x\in S$ and $x\notin\bigcup\{U : U\in\mathcal{U}\}\Leftrightarrow x\in S\setminus\bigcup\{U : U\in\mathcal{U}\}$

b) Y is closed $\Leftrightarrow X\setminus Y$ is open
Y is open $\Leftrightarrow X\setminus Y$ closed

$\bigcap Y = X\setminus\bigcup\{X\setminus Y\}$

Since $X\setminus Y$ open $\forall\, Y$, $\forall\, p\in\bigcup\{X\setminus Y\}$, $\exists\, B_r(p)\in X\setminus Y$ for whichever Y that $p\in X\setminus Y$
so $\bigcup\{X\setminus Y\}$ open.

so $X\setminus\bigcup\{X\setminus Y\}$ closed so $\bigcap Y$ closed.

13.7) If S is open, $\forall p \in S$, $\exists B_r(p) \subset S$.

For $p_1 \in S$ Let $B_1 = B_R$ st $R = \sup\{r : B_r(p) \subset S\}$

since ball is open $B_R \subset S$, but contains all points in S

Construct new B_2 for a point $p_2 \in S \setminus B_1$

Continue doing this possibly infinitely for all $p \in S$.

The union of these is disjoint and contains all $p \in S$

5) Let $A = \bigcap Y$ st $S \subset Y$ & $Y \subset X$

$\Rightarrow$: if $x \in \overline{S}$, $\exists (p_n)$ in S st $(p_n) \to x$

Then for any closed Y such that $S \subset Y$, there is (p_n) in Y which converges to x and Y is closed so $x \in Y$.

Thus $x \in A$.

$\Leftarrow$: if $x \in A$, since $\overline{S}$ is contains S and is closed (4) $x \in \overline{S}$.

$\therefore \overline{S} = A$ QED

4) Want to show $\overline{\overline{S}} = \overline{S} = \{p \in X \mid \exists (p_n) \in \overline{S} \text{ st } p_n \to p\}$

Clearly $\overline{S} \subset \overline{\overline{S}}$ because can take a sequence of desired point in $\overline{S}$.

Need to show: If $\exists (p_n) \to p$ w/ $(p_n) \in \overline{S}$, then $\exists (q_n) \in S$, $(q_n) \to p$

Since $\forall p_i \in (p_n)$, $\exists (a_n) \to p_i$,

take

$a_{11} \ \cdots \ a_{1n} \to p_1$

$a_{21} \ \cdots \ a_{2n} \to p_2$

By Cantor's diagonal argument, this sequence in $\overline{S} \to p$

So $\overline{\overline{S}} \subset \overline{S}$ so $\overline{S} = \overline{\overline{S}}$