13.3, 13.5, 13.7

13.3) a) $d(x,y)=0 \Leftrightarrow \sup\{|x_i-y_i| : i=1,2,\dots\}=0 \Leftrightarrow |x_i-y_i|=0 \ \forall\ i=1,2,\dots$

$\Leftrightarrow x=y$ ✓

$d(y,x)=\sup\{|y_i-x_i| : i=1,2,\dots\}= \sup\{|x_i-y_i| \mid i=1,2,\dots\}=d(x,y)$ ✓

$d(x,y)=\sup\{|x_i-y_i| : i=1,2,\dots\}= \sup\{K : K\geq 0\}\geq 0$ ✓

$d(x,y)+d(y,z)=\sup\{|x_i-y_i|\}+\sup\{|y_j-z_j|\}$

if sup is for $i=j$ then $d(x,y)+d(y,z)=d(x,z)$
else $d(x,y)+d(y,z)>d(x,z)$

b) No. Metrics must be finite but if $x=(1,1,1,\dots)$ and $y=(0,0,0,\dots)$ then the sum is infinite.

13.5) a) $x\in\bigcap\{S\setminus U : U\in\mathcal{U}\} \Leftrightarrow \forall\ U\in\mathcal{U},\ x\in S\setminus U$
$\Leftrightarrow x\in S$ and $x\notin U\ \forall\ U\in\mathcal{U}$
$\Leftrightarrow x\in S$ and $x\notin\bigcup\{U : U\in\mathcal{U}\} \Leftrightarrow x\in S\setminus\bigcup\{U : U\in\mathcal{U}\}$

b) Y is closed $\Leftrightarrow X\setminus Y$ is open
Y is open $\Leftrightarrow X\setminus Y$ closed

$\bigcap Y = X\setminus\bigcup_{\forall Y}\{X\setminus Y\}$

Since $X\setminus Y$ open, $\forall p\in\bigcup\{X\setminus Y\}$, $\exists B_r(p)\in X\setminus Y$ for whichever Y that $p\in X\setminus Y$
so $\bigcup\{X\setminus Y\}$ open.

so $X\setminus\bigcup\{X\setminus Y\}$ closed so $\bigcap Y$ closed.

13.7) If S is open, $\forall p \in S$, $\exists B_r(p) \subset S$.

For $p_i \in S$ Let $B_i = B_R$ st $R = \sup\{r : B_r(p) \subset S\}$

since ball is open $B_R \subset S$, but contains all points in S

Construct new B_2 for a point $p_2 \in S \setminus B_1$

Continue doing this possibly infinitely for all $p \in S$.

The union of these is disjoint and contains all $p \in S$

5) Let $A = \bigcap Y$ st $S \subset Y$ & $Y \subset X$

$\Rightarrow$: if $x \in \bar{S}$, $\exists (p_n)$ in S st $(p_n) \to x$

Then for any closed Y such that $S \subset Y$, there is (p_n) in Y which converges to x and Y is closed so $x \in Y$.

Thus $x \in A$.

$\Leftarrow$: if $x \in A$, since $\bar{S}$ is contains S and is closed (Y) $x \in \bar{S}$.

$\therefore \bar{S} = A$ QED

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4) Want to show $\bar{\bar{S}} = \bar{S} = \{p \in X \mid \exists (p_n) \in \bar{S} \text{ st } p_n \to p\}$

Clearly $\bar{S} \subset \bar{\bar{S}}$ because can take a sequence of desired point in $\bar{S}$.

Need to show: If $\exists (p_n) \to p$ w/ $(p_n) \in \bar{S}$, then $\exists (q_n) \in S$, $(q_n) \to p$

Since $\forall p_i \in (p_n)$, $\exists (a_n) \to p_i$,

take

$$\begin{array}{l} a_{11} \text{———} a_{1n} \cdots \to p_1 \\ a_{21} \text{———} a_{2n} \cdots \to p_2 \\ \vdots \end{array}$$

By Cantor's diagonal argument, this sequence in $\bar{S} \to p$

So $\bar{\bar{S}} \subset \bar{S}$ so $\bar{S} = \bar{\bar{S}}$